Edgeworth Equilibria: Separable and Non-separable Commodity Spaces

Anuj Bhowmik

Economic Research Unit
Indian Statistical Institute
203 Barackpore Trunk Road
Kolkata 700108
India

Email: anujbhowmik09@gmail.com

5th CMSS Summer Workshop
University of Auckland, 11th December 2013
1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When $\text{int} Y_+ = \emptyset$ and $Y$ is Separable
4. The Case When $\text{int} Y_+ \neq \emptyset$ and $Y$ is Non-separable
5. Conclusion
Outline

1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When $\text{int} Y_+ = \emptyset$ and $Y$ is Separable
4. The Case When $\text{int} Y_+ \neq \emptyset$ and $Y$ is Non-separable
5. Conclusion
1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When \( \text{int} Y_+ = \emptyset \) and \( Y \) is Separable
4. The Case When \( \text{int} Y_+ \neq \emptyset \) and \( Y \) is Non-separable
5. Conclusion
1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When $\text{int } Y_+ = \emptyset$ and $Y$ is Separable
4. The Case When $\text{int } Y_+ \neq \emptyset$ and $Y$ is Non-separable
5. Conclusion
Outline

1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When \( \text{int} Y_+ = \emptyset \) and \( Y \) is Separable
4. The Case When \( \text{int} Y_+ \neq \emptyset \) and \( Y \) is Non-separable
5. Conclusion
Banach Lattice

Let $X$ be a vector space. If $\geq$ is a partial order on $X$, then the pair $(X, \geq)$ is called an **ordered vector space** whenever for any $x, y, z \in X$ and any positive real number $\alpha$, $x \geq y$ implies that $\alpha x + z \geq \alpha y + z$.

Recall that a **Riesz space** is an ordered vector space that is also a lattice.

A function $\| \cdot \| : X \to \mathbb{R}_+$ is called a **norm** on $X$ if

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$; and
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. 

**Edgeworth Equilibria: Separable and Non-separable Commodity Spaces**
For any element $x$ of a Riesz space, $|x|$ stands for the absolute value of $x$ and is defined by $|x| = x^+ + x^-$, where

$$x^+ = x \lor 0 \text{ and } x^- = (-x) \lor 0.$$ 

A norm is called a lattice norm if $|x| < |y|$ implies $\|x\| \leq \|y\|$. A normed Riesz space is a Riesz space with a lattice norm.

A complete normed Riesz space is called a Banach lattice.

A Banach lattice $X$ is separable if there exists a countable dense subset $D$ of $X$. 

Edgeworth Equilibria: Separable and Non-separable Commodity Spaces
Infinite Dimensional commodity spaces arise in economies if one considers infinite time horizon, infinitely many states of nature, and infinitely many variations in any of the characteristic describing commodities.

Some Examples

(i) $\ell_\infty$: the space of real bounded sequences with the supremum norm

$$\| \{x_n : n \geq 1\} \|_\infty = \sup \{|x_n| : n \geq 1\}.$$  

(ii) $\ell_p$: the space of real sequences $\{x_n : n \geq 1\}$ equipped with the norm

$$\| \{x_n : n \geq 1\} \|_p = \left( \sum_{n \geq 1} |x_n|^p \right)^{1/p},$$

where $1 \leq p < \infty$.

(iii) $C[a, b]$: the space of real-valued continuous functions on a closed interval $[a, b]$ with the supremum norm

$$\| f \|_\infty = \sup \{|f(x)| : x \in [a, b]\}.$$
The purpose of this presentation is to show an extension of the equivalence result between the core and the set of Walrasian allocations in an economy with an atomless measure space of agents and a Banach lattice as the commodity space when agents use their private information.

Some initial results in a deterministic economy can be found in Aumann (1964), Hildenbrand (1974), Rustichini and Yannelis (1991), and Shitovitz (1973).
We consider a model of a pure exchange economy $\mathcal{E}$ with differential information.

**Description of the Economic Model**

The space of state nature is a measurable space $(\Omega, \mathcal{F})$, where $\Omega$ is a finite set containing $m$ elements.

The space of agents is a measure space $(T, \Sigma, \mu)$ with an atomless complete finite positive measure $\mu$.

The commodity space is a Banach lattice $Y$, and the positive cone $Y_+$ is the consumption set for each agent $t \in T$ in every state of nature $\omega \in \Omega$.

Each agent $t \in T$ has $(\mathcal{F}_t, U_t, a(t, \cdot), \mathbb{Q}_t)$ as characteristics.
We consider a model of a pure exchange economy $\mathcal{E}$ with differential information.

**Description of the Economic Model**

The **space of state nature** is a measurable space $(\Omega, \mathcal{F})$, where $\Omega$ is a finite set containing $m$ elements.

The **space of agents** is a measure space $(T, \Sigma, \mu)$ with an atomless complete finite positive measure $\mu$.

The **commodity space** is a Banach lattice $Y$, and the positive cone $Y_+$ is the **consumption set** for each agent $t \in T$ in every state of nature $\omega \in \Omega$.

Each agent $t \in T$ has $(\mathcal{F}_t, U_t, a(t, \cdot), Q_t)$ as characteristics.
Agent’s Characteristics

- $\mathcal{F}_t$ is the $\sigma$-algebra generated by a partition $\Pi_t$ of $\Omega$, representing the *private information*.
- $U_t : \Omega \times Y_+ \to \mathbb{R}$ is a *random utility function* of $t$.
- $a(t, \cdot) : \Omega \to Y_+$ is the *random initial endowment* of $t$.
- $Q_t$ is a probability measure on $\Omega$, giving the *prior belief* of $t$.

The economy extends over two time periods $\tau = 0, 1$. Consumption takes place at $\tau = 1$. At $\tau = 0$, there is uncertainty over the states and agents make contracts that are contingent on the realized state at $\tau = 1$. 
The Economic Model Continued

**Agent’s Characteristics**

- $\mathcal{F}_t$ is the $\sigma$-algebra generated by a partition $\Pi_t$ of $\Omega$, representing the *private information*.
- $U_t : \Omega \times Y_+ \to \mathbb{R}$ is a *random utility function* of $t$.
- $a(t, \cdot) : \Omega \to Y_+$ is the *random initial endowment* of $t$.
- $Q_t$ is a probability measure on $\Omega$, giving the *prior belief* of $t$.

The economy extends over two time periods $\tau = 0, 1$. Consumption takes place at $\tau = 1$. At $\tau = 0$, there is uncertainty over the states and agents make contracts that are contingent on the realized state at $\tau = 1$. 
A function $f : T \times \Omega \to Y_+$ is said to be an *allocation* if $f(\cdot, \omega)$ is Bochner integrable for all $\omega \in \Omega$ and $f(t, \cdot) \in L_t \mu$-a.e., where

$$L_t = \left\{ x \in (Y_+)^\Omega : x \text{ is } \mathcal{F}_t\text{-measurable} \right\}.$$  

It is assumed that $a$ is an allocation and $a(t, \omega)$ is a quasi-interior point of $Y_+$ for all $(t, \omega) \in T \times \Omega$.

It is said to be *$S$-feasible* (resp. *$S$-exactly feasible*) if for all $\omega \in \Omega$,

$$\int_S f(\cdot, \omega) d\mu \leq (=) \int_S a(\cdot, \omega) d\mu.$$

If $f$ is *$T$-feasible* (resp. *$T$-exactly feasible*) then it is simply termed as *feasible* (resp. *exactly feasible*).
For any \( n \geq 1 \), the \((n - 1)\)-simplex of \( \mathbb{R}^n \) is defined as

\[
\Delta^n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}.
\]

Consider a function \( \varphi : (T, \Sigma, \mu) \to \Delta^m \) defined by \( \varphi(t) = \mathcal{Q}_t \) for all \( t \in T \). For each \( \omega \in \Omega \), define a function \( \psi_\omega : T \times Y_+ \to \mathbb{R} \) by \( \psi_\omega(t, x) = U_t(\omega, x) \).

\((A_1)\) For each \( (t, \omega) \in T \times \Omega \), \( U_t(\omega, \cdot) : Y_+ \to \mathbb{R} \) is strictly monotone.

\((A_2)\) The function \( \varphi \) is measurable, where \( \Delta^m \) is endowed with the Borel structure.

\((A_3)\) For each \( \omega \in \Omega \), the function \( \psi_\omega \) is Carathéodory.
The *ex ante expected utility* of an agent $t$ for a given random consumption bundle $x : \Omega \to Y_+$ is defined by

$$E^{Q_t}(U_t(\cdot, x(\cdot))) = \sum_{\omega \in \Omega} U_t(\omega, x(\omega))Q_t(\omega).$$

A *price system* is a non-zero function $p : \Omega \to Y^*$, where $Y^*$ is the positive cone of the norm-dual $Y^*$ of $Y$.

The *budget set* of agent $t$ with respect to a price system $p$ is defined by

$$B_t(p) = \{ x \in L_t : E[\langle x, p \rangle] \leq E[\langle a(t, \cdot), p \rangle] \},$$

where for all $x \in Y_+^\Omega$,

$$E[\langle x, p \rangle] = \sum_{\omega \in \Omega} \langle x(\omega), p(\omega) \rangle.$$
Equilibrium Notions

A *Walrasian expectations equilibrium* of $E$ is a pair $(f, p)$, where $f$ is a feasible allocation and $p$ is a price system such that

$$f(t, \cdot) \in \arg \max \left\{ \mathbb{E}^Q_t(x) : x \in B_t(p) \right\} \text{ } \mu\text{-a.e.},$$

and

$$\mathbb{E} \left[ \left\langle \int_T fd\mu, p \right\rangle \right] = \mathbb{E} \left[ \left\langle \int_T ad\mu, p \right\rangle \right].$$

In this case, $f$ is called a *Walrasian expectations allocation* and the set of such allocations is denoted by $W(E)$. 

Infinite dimensional Commodity Spaces
Economic Model
The Case When $\text{int} Y_+ = \emptyset$ and $Y$ is Separable
The Case When $\text{int} Y_+ \neq \emptyset$ and $Y$ is Non-separable
Conclusion

Edgeworth Equilibria: Separable and Non-separable Commodity Spaces
Equilibrium Notions Continued

An allocation $f$ is *privately blocked* by a coalition $S$ if there exists an $S$-feasible allocation $g$ such that

$$E^Q_t (g(t, \cdot)) > E^Q_t (f(t, \cdot)) \mu\text{-a.e. on } S.$$ 

The *private core* of $E$, denoted by $PC(E)$, is the set of feasible allocations which are not privately blocked by any coalition.

The Case When $\text{int} \, Y_+ \neq \emptyset$ and $Y$ is Separable


Example (Rustichini and Yannelis (1991))

Consider the deterministic economy

\[ \mathcal{E} = \left\{ (T, \Sigma, \mu); \ell_2^+; (U_t, a(t))_{t \in T} \right\}, \]

where

(i) \( T = [0, 1] \) and \( \Sigma \) is the \( \sigma \)-algebra of Lebesgue measurable subsets of \( T \) with the Lebesgue measure \( \mu \);

(ii) for all \( t \in T \),

\[ U_t(x) = \sum_{n \geq 1} \frac{1 - \exp(-n^2x_n)}{n^2} \quad \text{and} \quad a(t) = \left\{ \frac{1}{n^2} : n \geq 1 \right\}. \]

It can be shown that \( PC(\mathcal{E}) = \{a\} \) and \( W(\mathcal{E}) = \emptyset \).
Extremely Desirable Bundle

Let $\omega \in \Omega$, $v > 0$ and $U$ be an open convex solid neighborhood of 0 in $Y$. Suppose that $K$ is the open cone spanned by $v + U$. The bundle $v$ is called an extremely desirable bundle with respect to $U$ at state $\omega$ if $x \in Y_+$ and $y \in (K + x) \cap Y_+$ together imply $U_t(\omega, y) > U_t(\omega, x)$ $\mu$-a.e.
Infinite dimensional Commodity Spaces
Economic Model

The Case When \( \text{int}Y_+ = \emptyset \) and \( Y \) is Separable
The Case When \( \text{int}Y_+ \neq \emptyset \) and \( Y \) is Non-separable

Conclusion

Additional Assumptions

\((A_4)\) For each \( \omega \in \Omega \), there is a \( v(\omega) > 0 \) such that \( v(\omega) \) is an extremely desirable bundle with respect to some open convex solid neighborhood \( U(\omega) \) of 0 in \( Y \).

\((A_5)\) Suppose that \( \delta_1, \ldots, \delta_m \) are positive numbers with \( \sum_{i=1}^{m} \delta_i = 1 \). If \( x_i \in Y_+ \) and \( x_i \notin \delta_i U(\omega) \) for all \( 1 \leq i \leq m \), then \( \sum_{i=1}^{m} x_i \notin U(\omega) \).

Put

\[ U = \left( \frac{1}{m} \bigcap_{\omega \in \Omega} U(\omega) \right)^m. \]

Let \( C \) and \( C(\omega) \) be the open convex cones spanned by \( \sum_{\omega \in \Omega} v(\omega)1_{\Omega} + U \) and \( v(\omega) + U(\omega) \) respectively for all \( \omega \in \Omega \).
Key Result

For any partition $\mathcal{Q}$ of $\Omega$, let

$$T_\mathcal{Q} = \{ t \in T : \Pi_t = \mathcal{Q} \}$$

and $\mathcal{P}(S) = \{ \mathcal{Q} : \mu(S \cap T_\mathcal{Q}) > 0 \}$.

Lemma 1

Assume $(A_1)$-$(A_5)$ and that $f \in PC(E)$. Let $g : S \times \Omega \rightarrow Y_+$ be defined by $g(t, \omega) = y_i(\omega)$ if $(t, \omega) \in S_i \times \Omega$, where for each $1 \leq i \leq m$ there is some $\mathcal{Q} \in \mathcal{P}(S)$ such that $S_i \subseteq S \cap T_\mathcal{Q}$ and $\mu(S_i) = \eta$. If $g(t, \cdot) \in L_t$ and $E^{Q_t}(g(t, \cdot)) > E^{Q_t}(f(t, \cdot)) \mu$-a.e. on $S$, then

$$\int_S (g - b) d\mu \notin -C,$$

where

$$b = \sum_{i=1}^{m} \left( \frac{1}{\eta} \int_{S_i} a d\mu \right) \chi_{S_i}.$$
Put

\[ P_f(t) = \left\{ x \in L_t : E^{Q_t}(x) > E^{Q_t}(f(t, \cdot)) \right\}. \]

**Lemma 2**

Assume \((A_1)-(A_5)\). If the correspondence \( F : T \Rightarrow Y^Ω_+ \) is defined by \( F(t) = \{ x - a(t, \cdot) : x \in P_f(t) \} \cup \{ 0 \} \) for all \( t \in T \), then \( \text{cl} \int_T Fd\mu \cap -C = \emptyset \).

Applying Lemma 2 and standard arguments, one can show that \( \mathcal{W}(E) = \mathcal{PC}(E) \).

Let \(\{ (F_i, U_i, a_i, Q_i) : i \geq 1\}\) be the set of different characteristics available in \(\mathcal{E}\) and \(T_i\) be the set of agents in \(T\) having the same characteristics as \((F_i, U_i, a_i, Q_i)\). Suppose that \(T_i \in \Sigma\) for all \(i \geq 1\). For any allocation \(f\) in \(\mathcal{E}\), let \(\hat{f} = \Xi(f)\) be an allocation defined by

\[
\hat{f}(t, \omega) = \begin{cases} 
  f(t, \omega), & \text{if } (t, \omega) \in T_i \times \Omega, \mu(T_i) = 0; \\
  \frac{1}{\mu(T_i)} \int_{T_i} f(\cdot, \omega) d\mu, & \text{if } (t, \omega) \in T_i \times \Omega, \mu(T_i) > 0.
\end{cases}
\]
Suppose \((A_1)\) and that \(U_t\) is continuous and concave for each \(t \in T\).

1. If \(f \in PC(E)\), then \(E^Q_t(f(t, \cdot)) = E^Q_t(\hat{f}(t, \cdot)) \mu\text{-a.e.}\)

2. Then \(\mathcal{W}(E) = PC(E)\).
Veto mechanisms

Suppose that $\mathcal{E}$ has finitely many agents, denoted by $N = \{1, \cdots, n\}$.

An allocation $x$ in $\mathcal{E}$ is called **Aubin dominated** if for all $i \in N$, there are $\alpha_i \in (0, 1]$ and $y_i \in L_i$ such that $E^Q_i(y_i) > E^Q_i(x_i)$ and

$$\sum_{i \in N} \alpha_i y_i(\omega) \leq \sum_{i \in N} \alpha_i a_i(\omega)$$

for all $\omega \in \Omega$.

An allocation $x \in \mathcal{W}(\mathcal{E})$ if and only if $x$ is not Aubin dominated.
For an allocation \( x \) in \( \mathcal{E} \) and a vector \( r = (r_1, \ldots, r_n) \in [0, 1]^n \), consider an asymmetric information economy \( \mathcal{E}(r, x) \) which is identical with \( \mathcal{E} \) except for the random initial endowment of each agent \( i \) being

\[
a_i(r_i, x_i) = r_i a_i + (1 - r_i)x_i.
\]

An allocation \( x \in \mathcal{W}(\mathcal{E}) \) if and only if \( x \) is not privately blocked by the grand coalition in every economy \( \mathcal{E}(r, x) \).
Thank You Very Much!