Edgeworth Equilibria: Separable and Non-separable Commodity Spaces

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Outline

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1. Infinite dimensional Commodity Spaces
2. Economic Model
3. The Case When $\operatorname{int} Y_+ = \emptyset$ and Y is Separable
4. The Case When $\operatorname{int} Y_+ \neq \emptyset$ and Y is Non-separable
5. Conclusion

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Banach Lattice

Let X be a vector space. If $\geq$ is a partial order on X, then the pair $(X, \geq)$ is called an *ordered vector space* whenever for any $x, y, z \in X$ and any positive real number α, $x \geq y$ implies that $\alpha x + z \geq \alpha y + z$.

Recall that a *Riesz space* is an ordered vector space that is also a lattice.

A function $\|\cdot\| : X \to \mathbb{R}_+$ is called a *norm* on X if

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$; and

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Banach Lattice (Continued)

For any element x of a Riesz space, $|x|$ stands for the *absolute value* of x and is defined by $|x| = x^+ + x^-$, where

$$x^+ = x \vee 0 \text{ and } x^- = (-x) \vee 0.$$

A norm is called a *lattice norm* if $|x| < |y|$ implies $\|x\| \leq \|y\|$. A *normed Riesz space* is a Riesz space with a lattice norm.

A complete normed Riesz space is called a *Banach lattice*.

A Banach lattice X is *separable* if there exists a countable dense subset D of X.

Non-isomorphic Commodity Spaces

Infinite Dimensional commodity spaces arise in economies if one considers infinite time horizon, infinitely many states of nature, and infinitely many variations in any of the characteristic describing commodities.

Infinite-dimensional commodity spaces have become well-established in the literature since the work of Debreu (1954), Peleg and Yaari (1970), Bewley (1972,1973), Kreps (1981), Jones (1983, 1984), Mas-Colell (1975, 1986).

Some Examples

(**i**) ℓ_∞: the space of real bounded sequences with the supremum norm

$$\|\{x_n : n \geq 1\}\|_\infty = \sup\{|x_n| : n \geq 1\}.$$

(**ii**) ℓ_p: the space of real sequences $\{x_n : n \geq 1\}$ equipped with the norm

$$\|\{x_n : n \geq 1\}\|_p = \left(\sum_{n \geq 1} |x_n|^p\right)^{\frac{1}{p}},$$

where $1 \leq p < \infty$.

(**iii**) $C[a, b]$: the space of real-valued continuous functions on a closed interval $[a, b]$ with the supremum norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\}.$$

The purpose of this presentation is to show an extension of the equivalence result between the core and the set of Walrasian allocations in an economy with an atomless measure space of agents and a Banach lattice as the commodity space when agents use their private information.

Some initial results in a deterministic economy can be found in Aumann (1964), Hildenbrand (1974), Rustichini and Yannelis (1991), and Shitovitz (1973).

Economic Model

We consider a model of a pure exchange economy $\mathscr{E}$ with differential information.

Description of the Economic Model

The space of state nature is a measurable space $(\Omega, \mathscr{F})$, where Ω is a finite set containing m elements.

The space of agents is a measure space (T, Σ, μ) with an atomless complete finite positive measure μ.

The commodity space is a Banach lattice Y, and the positive cone Y_+ is the consumption set for each agent $t \in T$ in every state of nature $\omega \in \Omega$.

Each agent $t \in T$ has $(\mathscr{F}_t, U_t, a(t, \cdot), \mathbb{Q}_t)$ as characteristics.

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The Economic Model Continued

Agent's Characteristics

- $\mathscr{F}_t$ is the σ-algebra generated by a partition Π_t of Ω, representing the *private information*.
- $U_t : \Omega \times Y_+ \to \mathbb{R}$ is a *random utility function* of t.
- $a(t, \cdot) : \Omega \to Y_+$ is the *random initial endowment* of t.
- $\mathbb{Q}_t$ is a probability measure on Ω, giving the *prior belief* of t.

The economy extends over two time periods $\tau = 0, 1$. Consumption takes place at $\tau = 1$. At $\tau = 0$, there is uncertainty over the states and agents make contracts that are contingent on the realized state at $\tau = 1$.

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Allocation and Feasibility

A function $f : T \times \Omega \to Y_+$ is said to be an *allocation* if $f(\cdot, \omega)$ is Bochner integrable for all $\omega \in \Omega$ and $f(t, \cdot) \in L_t$ μ-a.e., where

$$L_t = \left\{ x \in (Y_+)^{\Omega} : x \text{ is } \mathscr{F}_t\text{-measurable} \right\}.$$

It is assumed that a is an allocation and $a(t, \omega)$ is a quasi-interior point of Y_+ for all $(t, \omega) \in T \times \Omega$.

It is said to be *S-feasible* (resp. *S-exactly feasible*) if for all $\omega \in \Omega$,

$$\int_S f(\cdot, \omega) d\mu \leq (=) \int_S a(\cdot, \omega) d\mu.$$

If f is T-feasible (resp. T-exactly feasible) then it is simply termed as feasible (resp. exactly feasible).

Basic Assumptions

For any $n \geq 1$, the $(n-1)$-simplex of $\mathbb{R}^n$ is defined as

$$\Delta^n = \left\{ x = (x_1, ..., x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}.$$

Consider a function $\varphi : (T, \Sigma, \mu) \to \Delta^m$ defined by $\varphi(t) = \mathbb{Q}_t$ for all $t \in T$. For each $\omega \in \Omega$, define a function $\psi_\omega : T \times Y_+ \to \mathbb{R}$ by $\psi_\omega(t, x) = U_t(\omega, x)$.

(**A**$_1$) For each $(t, \omega) \in T \times \Omega$, $U_t(\omega, \cdot) : Y_+ \to \mathbb{R}$ is strictly monotone.

(**A**$_2$) The function φ is measurable, where Δ^m is endowed with the Borel structure.

(**A**$_3$) For each $\omega \in \Omega$, the function ψ_ω is Carathéodory.

Some Concepts

The *ex ante expected utility* of an agent t for a given random consumption bundle $x : \Omega \to Y_+$ is defined by

$$\mathbb{E}^{\mathbb{Q}_t}(U_t(\cdot, x(\cdot))) = \sum_{\omega \in \Omega} U_t(\omega, x(\omega))\mathbb{Q}_t(\omega).$$

A *price system* is a non-zero function $p : \Omega \to Y_+^*$, where Y_+^* is the positive cone of the norm-dual Y^* of Y.

The *budget set* of agent t with respect to a price system p is defined by

$$B_t(p) = \{x \in L_t : \mathbb{E}[\langle x, p \rangle] \leq \mathbb{E}[\langle a(t, \cdot), p \rangle]\},$$

where for all $x \in Y_+^\Omega$,

$$E[\langle x, p \rangle] = \sum_{\omega \in \Omega} \langle x(\omega), p(\omega) \rangle.$$

Equilibrium Notions

A *Walrasian expectations equilibrium* of $\mathscr{E}$ is a pair (f, p), where f is a feasible allocation and p is a price system such that

$$f(t, \cdot) \in \operatorname{argmax} \left\{ \mathbb{E}^{\mathbb{Q}_t}(x) : x \in B_t(p) \right\} \ \mu\text{-a.e.},$$

and

$$\mathbb{E}\left[\left\langle \int_T f d\mu, p \right\rangle\right] = \mathbb{E}\left[\left\langle \int_T a d\mu, p \right\rangle\right].$$

In this case, f is called a *Walrasian expectations allocation* and the set of such allocations is denoted by $\mathscr{W}(\mathscr{E})$.

Equilibrium Notions Continued

An allocation f is *privately blocked* by a coalition S if there exists an S-feasible allocation g such that

$$\mathbb{E}^{\mathbb{Q}_t}(g(t,\cdot)) > \mathbb{E}^{\mathbb{Q}_t}(f(t,\cdot)) \ \mu\text{-a.e. on } S.$$

The *private core* of $\mathscr{E}$, denoted by $\mathscr{PC}(\mathscr{E})$, is the set of feasible allocations which are not privately blocked by any coalition.

The Case When $\operatorname{int} Y_+ \neq \emptyset$ and Y is Separable

[1] E. Einy, D. Moreno, B. Shitovitz, Competitive and core allocations in large economies with differential information, Econ. Theory **18** (2001), 321– 332.

[2] Ö. Evren, F. Hüsseinov, Theorems on the core of an economy with infinitely many commodities and consumers, J. Math. Econ. **44** (2008), 1180–1196.

Example (Rustichini and Yannelis (1991))

Consider the deterministic economy

$$\mathscr{E} = \left\{ (T, \Sigma, \mu); \ell_2^+; (U_t, a(t))_{t \in T} \right\},$$

where

(i) $T = [0, 1]$ and Σ is the σ-algebra of Lebesgue measurable subsets of T with the Lebesgue measure μ;

(ii) for all $t \in T$,

$$U_t(x) = \sum_{n \geq 1} \frac{(1 - \exp(-n^2 x_n))}{n^2} \text{ and } a(t) = \left\{ \frac{1}{n^2} : n \geq 1 \right\}.$$

It can be shown that $\mathscr{PC}(\mathscr{E}) = \{a\}$ and $\mathscr{W}(\mathscr{E}) = \emptyset$.

Extremely Desirable Bundle

Let $\omega \in \Omega$, $v > 0$ and U be an open convex solid neighborhood of 0 in Y. Suppose that K is the open cone spanned by $v + U$. The bundle v is called an *extremely desirable bundle* with respect to U at state ω if $x \in Y_+$ and $y \in (K + x) \cap Y_+$ together imply $U_t(\omega, y) > U_t(\omega, x)$ μ-a.e.

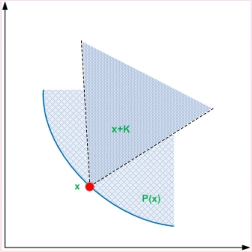

Figure : Extremely Desirable Bundle

Additional Assumptions

(**A**$_4$) For each $\omega \in \Omega$, there is a $v(\omega) > 0$ such that $v(\omega)$ is an extremely desirable bundle with respect to some open convex solid neighborhood $U(\omega)$ of 0 in Y.

(**A**$_5$) Suppose that $\delta_1, ..., \delta_m$ are positive numbers with $\sum_{i=1}^m \delta_i = 1$. If $x_i \in Y_+$ and $x_i \notin \delta_i U(\omega)$ for all $1 \leq i \leq m$, then $\sum_{i=1}^m x_i \notin U(\omega)$.

Put

$$U = \left(\frac{1}{m} \bigcap_{\omega \in \Omega} U(\omega) \right)^m .$$

Let C and $C(\omega)$ be the open convex cones spanned by $\sum_{\omega \in \Omega} v(\omega) \mathbf{1}_\Omega + U$ and $v(\omega) + U(\omega)$ respectively for all $\omega \in \Omega$.

Key Result

For any partition $\mathscr{Q}$ of Ω, let

$$T_{\mathscr{Q}} = \{t \in T : \Pi_t = \mathscr{Q}\} \text{ and } \mathfrak{P}(S) = \{\mathscr{Q} : \mu(S \cap T_{\mathscr{Q}}) > 0\}.$$

Lemma 1

Assume $(\mathbf{A}_1)$-$(\mathbf{A}_5)$ and that $f \in \mathscr{PC}(\mathscr{E})$. Let $g : S \times \Omega \to Y_+$ be defined by $g(t,\omega) = y_i(\omega)$ if $(t,\omega) \in S_i \times \Omega$, where for each $1 \leq i \leq m$ there is some $\mathscr{Q} \in \mathfrak{P}(S)$ such that $S_i \subseteq S \cap T_{\mathscr{Q}}$ and $\mu(S_i) = \eta$. If $g(t,\cdot) \in L_t$ and $\mathbb{E}^{\mathbb{Q}_t}(g(t,\cdot)) > \mathbb{E}^{\mathbb{Q}_t}(f(t,\cdot))$ μ-a.e. on S, then $\int_S (g-b)d\mu \notin -C$, where

$$b = \sum_{i=1}^{m} \left(\frac{1}{\eta} \int_{S_i} a d\mu \right) \chi_{S_i}.$$

Put

$$P_f(t) = \left\{ x \in L_t : \mathbb{E}^{\mathbb{Q}_t}(x) > \mathbb{E}^{\mathbb{Q}_t}(f(t,\cdot)) \right\}.$$

Lemma 2

Assume $(\mathbf{A}_1)$-$(\mathbf{A}_5)$. If the correspondence $F : T \rightrightarrows Y_+^{\Omega}$ is defined by $F(t) = \{x - a(t,\cdot) : x \in P_f(t)\} \cup \{0\}$ for all $t \in T$, then $\operatorname{cl} \int_T F d\mu \cap -C = \emptyset$.

Applying Lemma 2 and standard arguments, one can show that $\mathscr{W}(\mathscr{E}) = \mathscr{PC}(\mathscr{E})$.

[3] K. Podczeck, Core and Walrasian equilibria when agents' characteristics are extremely dispersed, Econ. Theory **22** (2003), 699–725.

[4] R. Tourky, N. C. Yannelis, Markets with many more agents than commodities: Aumann's "hidden" assumption, J. Econ. Theory **101** (2001), 189–221.

Let $\{(\mathscr{F}_i, U_i, a_i, \mathbb{Q}_i) : i \geq 1\}$ be the set of different characteristics available in $\mathscr{E}$ and T_i be the set of agents in T having the same characteristics as $(\mathscr{F}_i, U_i, a_i, \mathbb{Q}_i)$. Suppose that $T_i \in \Sigma$ for all $i \geq 1$. For any allocation f in $\mathscr{E}$, let $\hat{f} = \Xi(f)$ be an allocation defined by

$$\hat{f}(t,\omega) = \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in T_i \times \Omega, \mu(T_i) = 0; \\ \frac{1}{\mu(T_i)} \int_{T_i} f(\cdot,\omega) d\mu, & \text{if } (t,\omega) \in T_i \times \Omega, \mu(T_i) > 0. \end{cases}$$

Suppose $(\mathbf{A}_1)$ and that U_t is continuous and concave for each $t \in T$.

(1) If $f \in \mathscr{P}\mathscr{C}(\mathscr{E})$, then $\mathbb{E}^{\mathbb{Q}_t}(f(t,\cdot)) = \mathbb{E}^{\mathbb{Q}_t}(\hat{f}(t,\cdot))$ μ-a.e.

(2) Then $\mathscr{W}(\mathscr{E}) = \mathscr{P}\mathscr{C}(\mathscr{E})$.

Veto mechanisms

Suppose that $\mathscr{E}$ has finitely many agents, denoted by $N = \{1, \cdots, n\}$.

An allocation x in $\mathscr{E}$ is called *Aubin dominated* if for all $i \in N$, there are $\alpha_i \in (0, 1]$ and $y_i \in L_i$ such that $\mathbb{E}^{\mathbb{Q}_i}(y_i) > \mathbb{E}^{\mathbb{Q}_i}(x_i)$ and

$$\sum_{i \in N} \alpha_i y_i(\omega) \leq \sum_{i \in N} \alpha_i a_i(\omega)$$

for all $\omega \in \Omega$.

An allocation $x \in \mathscr{W}(\mathscr{E})$ if and only if x is not Aubin dominated.

For an allocation x in $\mathscr{E}$ and a vector $r = (r_1, \cdots, r_n) \in [0, 1]^n$, consider an asymmetric information economy $\mathscr{E}(r, x)$ which is identical with $\mathscr{E}$ except for the random initial endowment of each agent i being

$$a_i(r_i, x_i) = r_i a_i + (1 - r_i)x_i.$$

An allocation $x \in \mathscr{W}(\mathscr{E})$ if and only if x is not privately blocked by the grand coalition in every economy $\mathscr{E}(r, x)$.

Thank You Very Much !