Path Independent Choice and the Ranking of Opportunity Sets

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3rd CMSS Workshop

Theory of Choice

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(CF2) For any $A, B \in 2^X$:

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- A path independent choice function is also called a *Plott function* (after Plott, 1973).

Non-Binary Choice

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- Path independence is **not** sufficient to ensure that the choice function is *binary*.
- A choice function $c : 2^X \rightarrow 2^X$ is *binary* if there exists a binary relation $\succsim \subseteq X \times X$ such that

$$c(A) = \max_{\succsim} A$$

where

$$\max_{\succsim} A \equiv \{x \in A \mid \text{there is no } y \in A \diagdown \{x\} \text{ with } y \succ x\}.$$

Non-Binary Choice

Example (Plott, 1973)

Let $X = \{x, y, z\}$ and define $c : 2^X \to 2^X$ as follows:

$$c(A) = \begin{cases} \{x, y\} & \text{if } A = X \\ A & \text{if } A \neq X \end{cases}$$

It is easily verified that c is a Plott function.
Since $c(X) = \{x, y\}$, we must have $x \succ z$ or $y \succ z$ if c is binary. But these contradict $c(\{x, z\}) = \{x, z\}$ and $c(\{y, z\}) = \{y, z\}$, respectively, so c is non-binary.

Ranking Opportunity Sets

Definition

An *opportunity set ranking* is a binary relation $\succsim^* \subseteq 2^X \times 2^X$ which is *reflexive*, *complete* and satisfies

$$A \succ^* \emptyset \quad \text{for all } A \neq \emptyset.$$

- Think of ranking restaurants (i.e., menus).

Ranking Opportunity Sets

- If (meal) choice is governed by the binary relation $\succsim \subseteq X \times X$, then opportunity sets (i.e., restaurants) are naturally ranked according to the following *indirect utility (IU)* principle:

$$A \succsim^* B \quad \Leftrightarrow \quad \left[\max_{\succsim} (A \cup B)\right] \cap A \neq \emptyset \tag{IU}$$

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- *What are the hallmarks of opportunity set rankings that obey the IU principle?*

Ranking Opportunity Sets

Theorem (Kreps, 1979)

The opportunity set ranking $\succsim^*$ satisfies (IU) for some complete, reflexive and transitive $\succsim \subseteq X \times X$ iff $\succsim^*$ is **transitive** *and satisfies*

$$A \succsim^* B \quad \Rightarrow \quad A \sim^* A \cup B \tag{K}$$

for every $A, B \subseteq X$.

Ranking Opportunity Sets

Definition (Lahiri, 2003)

An opportunity set ranking $\succsim^*$ is *justifiable* if it satisfies (IU) for some *complete* and *reflexive* (but not necessarily transitive) $\succsim \subseteq X \times X$

Ranking Opportunity Sets

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- The IU principle embodies two fundamental ideas:
  1. Consequentialism.
  2. Binariness.
- Many papers relax (1). We maintain (1) but relax (2).

Ranking Opportunity Sets

- By analogy with the IU condition

$$A \succsim^* B \quad \Leftrightarrow \quad \left[\max_{\succsim}(A \cup B)\right] \cap A \neq \emptyset,$$

we propose:

Definition

An opportunity set ranking $\succsim^*$ is *Plott consistent* if there exists a Plott function $c : 2^X \to 2^X$ such that

$$A \succsim^* B \quad \Leftrightarrow \quad c(A \cup B) \cap A \neq \emptyset$$

for any $A, B \subseteq X$.

Ranking Opportunity Sets

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1. Characterise the Plott consistent rankings.
2. Compare Plott consistency and justifiability.
3. Raise a question of interpretation and pose an open problem.

Plott Consistency

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Plott Consistency

1. *What are the necessary and sufficient conditions (on $\succsim^*$) for Plott consistency?*

- It is easy to verify that the Kreps condition

$$A \succsim^* B \quad \Rightarrow \quad A \sim^* A \cup B \tag{K}$$

  is necessary.
- However, transitivity of $\succsim^*$ is not:

Plott Consistency

Example (continued)

Recall that $X = \{x, y, z\}$ and

$$c(A) = \begin{cases} \{x, y\} & \text{if } A = X \\ A & \text{if } A \neq X \end{cases}$$

is a Plott function. Applying Plott consistency, we have: $\{x, y\} \sim^* \{y\}$ and $\{y\} \sim^* \{z\}$, but $\{x, y\} \succ^* \{z\}$.

Plott Consistency

Theorem

Given an opportunity set ranking $\succsim^*$, the following are equivalent:

(i) $\succsim^*$ *is Plott consistent.*

(ii) $\succsim^*$ *satisfies the following conditions for any $A, B, C \subseteq X$: the Kreps condition (K), plus*

$$B \succ^* A \quad \Rightarrow \quad B \cup C \succ^* A \;\; \text{and} \;\; B \succ^* A \setminus C \tag{SM}$$

and

$$[B \succ^* A \;\; \text{and} \;\; B \succ^* C] \;\Rightarrow\; B \succ^* A \cup C \tag{U}$$

- This result is “tight” in that none of (K), (SM) or (U) can be dropped without violating the equivalence.

Plott Consistency

- The proof of the theorem draws liberally on results from abstract convexity theory, and especially the papers by Aizerman and Malishevski (1981) and Danilov and Koshevoy (2006).

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- The proof of the theorem draws liberally on results from abstract convexity theory, and especially the papers by Aizerman and Malishevski (1981) and Danilov and Koshevoy (2006).
- Given an abstract convex geometry on X, consider the complete and reflexive binary relation $\succsim^* \subseteq 2^X \times 2^X$ defined as follows: $A \succ^* B$ iff all the extreme points of $A \cup B$ are contained in $A \setminus B$.

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- Neither are all justifiable rankings Plott consistent.

Plott Consistency and Justifiability

2. *What is the relationship between Plott consistency and Justifiability?*

- Not all Plott consistent rankings are justifiable – this can be proved using Plott's example.
  - Plott consistency permits fundamental non-binariness.
- Neither are all justifiable rankings Plott consistent.
  - Plott consistency imposes *quasi-transitivity* of $\succsim^*$.

Plott Consistency and Justifiability

Theorem

A justifiable opportunity set ranking $\succsim^*$ is Plott consistent iff it is quasi-transitive.

Plott Consistency and Justifiability

- An opportunity set ranking $\succsim^* \subseteq 2^X \times 2^X$ satisfies *Weak Expansion* if: for any $A, B \in \mathcal{B}$ and any $x \in X$,

$$\{x\} \succsim^* A \ \text{ and } \ \{x\} \succsim^* B \quad \Rightarrow \quad \{x\} \succsim^* A \cup B \tag{WE}$$

Theorem

A Plott consistent opportunity set ranking $\succsim^*$ is justifiable iff it satisfies Weak Expansion.

A question of interpretation

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- Non-binary choice implies *context-dependence*.
- Is Plott consistency appropriate for context-dependent choice?

A question of interpretation

Example (continued)

$X = \{x, y, z\}$ and

$$c(A) = \begin{cases} \{x, y\} & \text{if } A = X \\ A & \text{if } A \neq X \end{cases}$$

Consider the following two-player game:

	x	y	z
α	$-$, 3	$-$, 0	$-$, 1
β	$-$, 0	$-$, 3	$-$, 1

Then c corresponds to choosing *undominated* strategies.

- Can we rule out $\{x, y\} \succ^* \{x\}$?

A question of interpretation

- Given an opportunity set ranking $\succsim^*$, define

$$\mathfrak{c}(A) = \bigcap \{B \subseteq A \mid B \succ^* A \setminus B\}$$

for each $A \subseteq X$.

Definition

Say that $\succsim^*$ is *strongly consequentialist* if $\mathfrak{c}(A) \neq \emptyset$ for every non-empty $A \subseteq X$, and

$$A \succsim^* B \quad \text{iff} \quad \mathfrak{c}(A \cup B) \cap A \neq \emptyset.$$

In this case, we call $\mathfrak{c} : 2^X \to 2^X$ the *revealed choice function* for $\succsim^*$.

A question of interpretation

Theorem

An opportunity set ranking $\succsim^* \subseteq 2^X \times 2^X$ *is strongly consequentialist iff*

…???

- Plott consistency is sufficient but not necessary.

A question of interpretation

Theorem

A strongly consequentialist opportunity set ranking $\succsim^* \subseteq 2^X \times 2^X$ *is Plott consistent iff*

...???