Geometry of distance-rationalization

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Outline

1. Motivation
2. Representation of anonymous and homogeneous rules
   - Compression of the data
   - The case of the l^1-votewise metrics
3. Geometric study of some properties
   - Discrimination
   - Other properties

Outline

Geometry of voting

Geometry is a classical approach in voting theory:

- H.P. Young. Social choice scoring functions (1975).
- D.G. Saari. Basic geometry of voting (1995).

Saari introduced the simplex representation.

General settings

- A list C of *candidates* and a list V of *voters*.
- Each voter has strict preferences over C.
- To any election $E =< C, V >$ is associated a *profile* corresponding to the list of the preferences of the voters.
- A *voting rule* associates to a profile a set of candidates.

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- A voting rule is defined by a couple (K, d),
  where d is a *distance* over elections and K is a *consensus*.
- The consensus represents the elections where the winner is "obvious".
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A meaningful framework

- Any rule is *distance-rationalizable*.
- A voting rule satisfies universality while a consensus satisfies uniqueness.
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To answer your question

- If a rule R is (K, d)-rationalizable, and if K' is an extension of K, then in the general case, R might not be (K', d)-rationalizable.
- It is true in most cases.

Outline

Embedding into $\mathbb{N}^{m!}$

A rule is *anonymous* if it only depends on the number of votes of each sort.

The set of profiles can be reduced to $\mathbb{N}^{m!}$.

Let $\mathcal{N}$ be the projection from $\mathscr{P}$ to $\mathbb{N}^{!m}$: $\mathcal{N}(\pi)_r = |\{v|\pi_v = r\}|$.

Anonymity

To be meaningful into $\mathbb{N}^{m!}$, a rule needs to verify for all profiles p and p'such that $\mathcal{N}(\pi) = \mathcal{N}(\pi')$, that $R(P) = R(P')$.

For an equivalence relation $\sim$, a *morphism* is a function f such that $x \sim y \implies f(x) = f(y)$.

We want our rules to be a morphism for the relation $x \sim_{\mathcal{N}} y \iff \mathcal{N}(x) = \mathcal{N}(y)$.

A rule is a morphism for $\sim_{\mathcal{N}}$ if and only if it is anonymous.

Anonymity for metrics

We want that if $x \sim x'$ and $y \sim y'$, then $d(x, y) = d(x', y')$.

It is the case for tournament metrics.

Not the case for votewise rules defined by EFS $\implies$ anonymization.

Example: N-votewise metric: $d'(\pi, \pi') = N(d(\pi_1, \pi'_1), \ldots, d(\pi_n, \pi'_n))$.

Idea: can we use $\min_\sigma d(\pi, \sigma(\pi'))$?

Anonymization of metrics

We define the quotient metric of d as

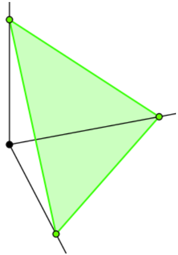
$$\delta(x, y) = \inf d(\pi_1, \pi'_1) + \dots d(\pi_k, \pi'_k),$$

where the infimum is taken over all finite sequences π, π' of elections such that $\mathcal{N}(\pi_1) = x$, $\mathcal{N}(\pi'_k) = y$ and for all i, $\pi_{i+1} \sim_{\mathcal{N}} \pi'_i$.

Indeed, for votewise metrics whose underlying norm is symmetric, $\delta(x, y) = \min_{\pi'} d(\pi, \pi') = \min_{\sigma} d(\pi, \sigma(\pi'))$.

Moreover, for these votewise metrics and anonymous consensus, the rationalization in $\mathbb{N}^{!m}$ with the quotient metric is equivalent to the usual rationalization.

Embedding into the simplex



The simplex represents the percentage distributions of the votes.

We define $\mathcal{D}(\pi) = \frac{|\{v|\pi_v = r\}|}{|V|}$.

$\mathcal{D}$ is a projection from the set of profiles to the rational points of the simplex.

Homogeneity

For a profile p, $p^{(k)}$ denotes the profile where each voter is split into k consecutive voters.

A rule is *homogeneous* if for all k and p, $R(p) = R(p^{(k)})$.

A rule is a morphism for $\sim_{\mathcal{D}}$ if and only if it is anonymous and homogeneous.

Homogeneity for metrics

We want that, for any k and k', $d(k\pi, k'\pi')$ stays the same.

We need to homogenize most of the metrics by dividing by the number of voters.

For example, as soon as the underlying norm is homogeneous, i.e. $N(x) = N(n^{(k)})$, a votewise metric is equivalent to its homogenized version.

Again, rationalizing with a metric or its anonymized and homogenized version is equivalent as soon as the consensus is anonymous and homogeneous.

Transportation metrics

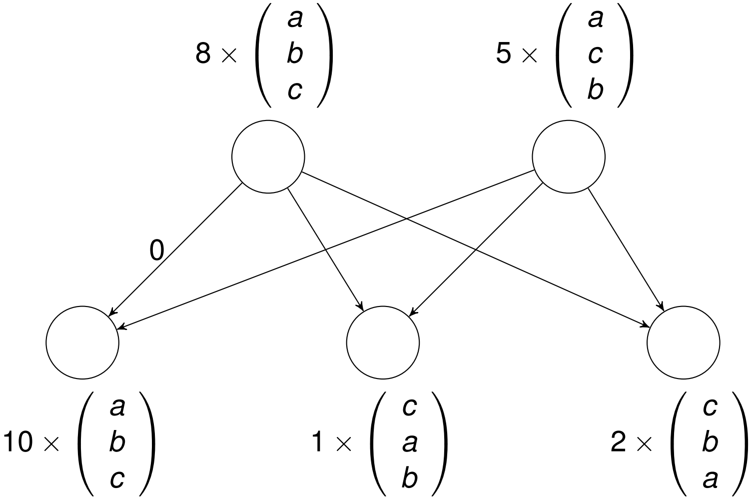
The l^p-transportation metric (also called Vasershtein metric) d_W^p, is defined by

$$d_W^p(x,y)^p = \min_A \sum_{r,r' \in S} A_{r,r'} d_S(r,r')^p,$$

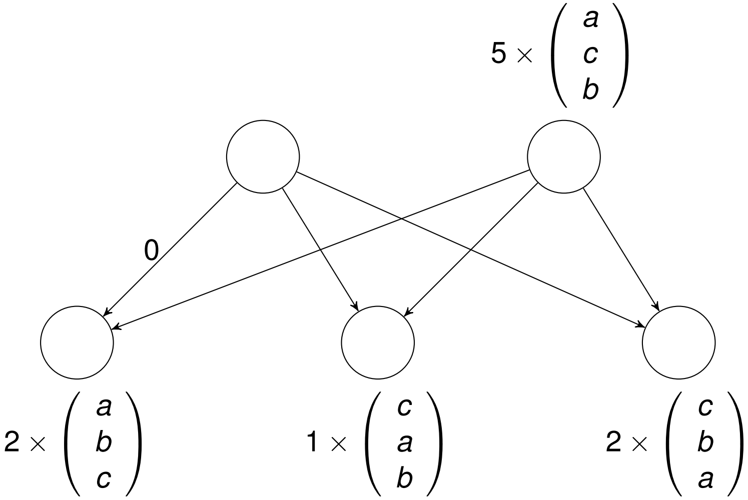
where the minimum is taken over all couplings of x and y, defined as nonnegative square matrices of size m! whose marginals are x and y respectively.

Basically, the Vasershtein metric gives the optimal cost of the transportation problem between x and y.

Equivalence to a transportation problem



Equivalence to a transportation problem (2)



Theorem 1

The metric induced by a l^p-votewise metric over the simplex is equal to the corresponding l^p-transportation metric.

Theorem 2

Any l^1-transportation metric induces a norm over the simplex.

Example

The Hamming metric induces the l^1-norm over the simplex.

Outline

Discrimination

A rule is *discriminating* if for any tied election, there exists an arbitrarily close untied election.

A *bisector* of two sets is the set of points equidistant to both sets.

In order to have a tie, we need to be in the bisector of two consensus sets.

Bisectors of two points

Theorem 3

If the unit ball of a Minkovski normed space is strictly convex, then all bisectors are homeomorphic to a hyperplane.
It is not the case if the ball is just convex: we can have *large* bisectors, i.e. bisectors containing a subspace of the same dimension as the space, or, alternatively, containing a point such that for a $\epsilon > 0$, the sphere of radius ϵ around 0 is contained into the bisector.

The example of the Manhattan norm

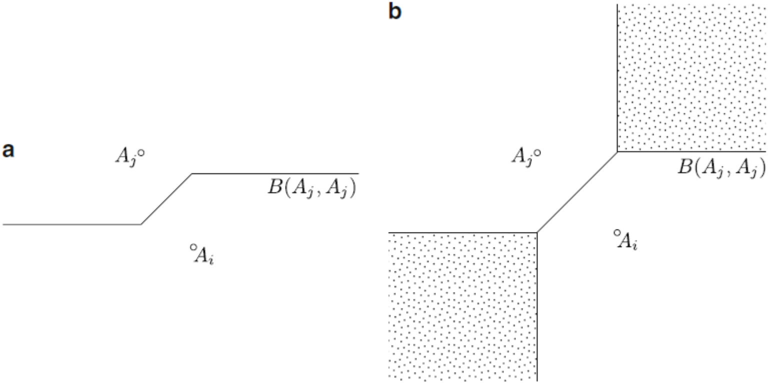


Fig. 1.3 Bisectors of the Manhattan norm for different conditions. (**a**) Bisector is piecewise linear curve. (**b**) Bisector contains subregions

Bisectors under the transportation metrics

We show that any 1-transportation metric can have large bisectors in the simplex:

Proposition

Let r, r_1, r_2 be rankings. We denote by d_1 and d_2 the distances $d(r, r_1)$ and $d(r, r_2)$. Let x and y be vectors $c + v$ and $c + w$ such that $v_r = -v_{r_1} = \frac{\epsilon}{d_1}$ and $w_r = -w_{r_2} = \frac{\epsilon}{d_2}$.
Then, for any point such that $z_r \geq 1 - (\frac{1}{m!} - \frac{\epsilon}{min(d_1, d_2)})$ is equidistant, according to the corresponding 1-transportation metric, from x and y.

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Bisectors under l^1

We now characterize the bisector for l^1:

Proposition

Let x and y be two points of $\mathbb{R}^n$. We denote by S the set of values $(x_i - y_i)$.
x and y have large bisectors under the Manhattan norm if and only if there exists a subset $S' \subset S$ such that $\sum_{e \in S'} e = \sum_{e \notin S'} e$.

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Bisectors of two hyperplanes

Proposition

Under the Manhattan norm, two distinct hyperplanes cannot have large bisectors.

Neutrality

A rule is *neutral* if it is invariant under permutation of the candidates.

With 3 candidates, we define the consensus set K such that any point of K^a is in the corresponding weak consensus or has proportion $\frac{1}{2}$ of the two rankings where a is ranked last.

Consistency

A rule is *consistent* if for any two ballots having the same winner, the union of the two ballots has the same winner.

It is equivalent to the fact that the sets where a candidate wins are convex.

Young's theorem: a rule which is anonymous, neutral and consistent is a scoring function.

Summary

- We introduced a general approach to study the distance-rationalization concept.
- The simplex seems a natural space to represent voting situations.
- We have studied the distance-rationalization of rules in the simplex.
- Outlook
  - A lot of different properties can be studied in this representation (neutrality, consistency, monotonicity...).
  - There are still a lot of questions about the general framework of distance-rationalization.