Freedom of Opportunity: Axiomatic Approaches

A Selective Survey

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  - Sometimes called *opportunity sets*.

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- Axioms are imposed on $\succsim$ which are consistent with this interpretation.
- Representations are derived:
  - equivalence of some tractable ranking rule (e.g., cardinality), possibly incorporating exogenous auxiliary information (e.g., indirect utility)
  - equivalent to the existence of an auxiliary structure (e.g., utility function on X) which generates the ranking according to a given rule (e.g., indirect utility).

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  - Are choices from opportunity sets mutually exclusive or not (*life courses* or *freedoms*)?
- Depending on how we interpret our task, there are different branches of the literature to follow.

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- The starting point is the axiomatisation of the cardinality ranking...

Extent of Freedom

Theorem (Pattanaik and Xu, 1990)

Given $\succsim \subseteq Z \times Z$*, the following are equivalent*

1. *The relation* $\succsim$ *is reflexive, transitive and satisfies, for every* $A, B \in Z$ *and every* $x, y \in X$ *and every* $z \in X \diagdown A \cup B$,

$$\{x\} \sim \{y\}$$

$$x \neq y \quad \Rightarrow \quad \{x, y\} \succ \{x\}$$

$$A \succsim B \quad \Leftrightarrow \quad A \cup \{z\} \succsim B \cup \{z\}$$

2. *For every* $A, B \in Z$

$$A \succsim B \quad \Leftrightarrow \quad \#A \geq \#B$$

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  - Indifference to shape-preserving transformations
  - Axiomatises ranking rules that combine cardinality and range (intersection or lexicographic combination)
  - Axioms are not as "basic" as one might like. For example:

$$x \in \mathrm{co}(A) \quad \Rightarrow \quad A \cup \{x\} \sim A$$

$$x \notin \mathrm{co}(A) \quad \Rightarrow \quad A \cup \{x\} \succ A$$

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  - A set $A \in Z$ is *homogeneous* if all elements are pairwise similar
  - A *similarity-based partition* of $A \in Z$ is a partition into homogeneous subsets. Thus, any refinement of a similarity-based partition is also a similarity-based partition

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  - A *similarity-based partition* of $A \in Z$ is a partition into homogeneous subsets. Thus, any refinement of a similarity-based partition is also a similarity-based partition
  - The value of $A \in Z$ is based on the number of cells in each maximally coarse similarity-based partition. (Pattanaik and Xu give a more precisely-stated rule and an axiomatisation of same.)

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  - Axiomatisation: for example, identifying basic freedom-improving operations.
  - Representation: obtaining convenient ranking conditions on pairs of distributions (analogous to SOSD, etc.)

Instrumental Value of Freedom

- The starting point is the *indirect utility* ranking based on some binary relation R on X:

$$A \succsim B \quad \Leftrightarrow \quad A \cap \max_R (A \cup B) \neq \emptyset$$

Instrumental Value of Freedom

Theorem (Kreps, 1979)

Given $\succsim \subseteq Z \times Z$, the following are equivalent

1. *The relation $\succsim$ is complete, transitive and satisfies, for every $A, B \in Z$,*
$$A \succsim B \quad \Rightarrow \quad A \sim A \cup B$$
2. *There exists a weak order R on X such that*
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  - Intersection and lexicographic combinations of cardinality and indirect utility (for given R)

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  - and with a dash of cardinality...Puppe and Xu (2010)

$$A \succsim B \quad \Leftrightarrow \quad \#\left[c\left(A \cup B\right) \cap A\right] \geq \#\left[c\left(A \cup B\right) \cap B\right]$$

where $c\left(E\right) = \left\{x \in E \mid E \succ E \setminus \{x\}\right\}$ are the "essential elements" of E.

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- ...or one may impute $\mathcal{R}$ from the axioms (Kreps, 1979; Nehring and Puppe, 1999)
- Some basic representations: intersections or unions of indirect utility orders, or weighted indirect utility

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  - If each R_k is a *linear* order on X, then c is the extreme point operator for some *abstract convex geometry (ACG)*
  - ACG's may be used to provide a generalisation of the Klemisch-Ahlert (1993) result – generalised notion of the "range" of opportunities

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- Given a closure operator σ, a set $A \subseteq X$ is said to be *closed* if $A = \sigma(A)$.
- Associated with any closure operator σ is an extreme point operator $c : 2^X \to 2^X$ defined as follows:

$$c(A) = \{x \in A \mid \sigma(A) \neq \sigma(A \setminus \{x\})\}.$$

Freedom and Convexity

- A closure operator is an *abstract convex geometry (ACG)* if it satisfies the following *anti-exchange property*: for any $A \subseteq X$ with $\sigma(A) = A$ and any distinct $x, y \in X \diagdown A$

$$y \in \sigma(A \cup \{x\}) \quad \Rightarrow \quad x \notin \sigma(A \cup \{y\})$$

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- If σ is an ACG, we refer to $\sigma(A)$ as the *convex hull* of A.

Freedom and Convexity

Theorem (Edelman and Jamison, 1985)

A closure operator $\sigma : 2^X \to 2^X$ *with associated extreme point operator* $c : 2^X \to 2^X$ *is an ACG iff* $\sigma(A) = \sigma(c(A))$ *for any* $A \subseteq X$.

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Theorem (Danilov, Koshevoy, Savaglio, 2012)

Given $\succsim \subseteq 2^X \times 2^X$, *the following are equivalent*

1. *The relation* $\succsim$ *is transitive and satisfies, for every* $A, B, C \in 2^X$,

$$A \subseteq B \quad \Rightarrow \quad B \succsim A \tag{MON}$$

$$A \succsim B \quad \Rightarrow \quad A \cup C \succsim B \cup C \tag{S-IND}$$

2. *There exists a closure operator* $\sigma : 2^X \to 2^X$ *such that*

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Theorem (Ryan, 2010; Danilov, Koshevoy, Savaglio, 2012)

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$$A \sim B \quad \Rightarrow \quad A \cap B \succsim A \cup B \qquad \text{(LE)}$$

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$$A \succsim B \text{ and } c(B) \subseteq A \quad \Rightarrow \quad A \cup C \succsim B \cup C \qquad \text{(E-S-IND)}$$

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    where $c(E) = \{x \in E \mid E \succ E \setminus \{x\}\}$.
    - Conjecture 3: Analogous extension of DKS result possible (using ideas from Kreps, 1979)
  - How should we strengthen (MON), (S-IND) and (LE) to ensure "representation" by a *convex shelling*? Can we determine the minimal dimension of such a "representation"?