Composition of Simple Games

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A simple game is a pair $G = (P_G, W_G)$, where $P_G$ is a set of players and $W_G \subseteq 2^{P_G}$ is a non-empty set of subsets (coalitions) which satisfy the monotonicity condition:

$$\text{if } X \in W_G \text{ and } X \subseteq Y, \text{ then } Y \in W_G$$

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$[39; 7, 7, 7, 7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
Consider the board of a large company, who vote to make strategy decisions under a certain voting rule. Suppose one of the board members retires, but it is decided that their knowledge and experience is too great to replace with just a single person. Instead, a group of people fills the one spot on the board. They collectively vote on each issue. A collective yes vote means that the ex-board members vote is a yes, a collective no means that the ex-board members vote is a no.
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What properties does the resulting voting structure (game) have?

Is it possible to incorporate all voters in a one-step process, or do we require two separate votes?
Definition

Let $G$ and $H$ be two games such that $P_G$ and $P_H$ are disjoint. Define the composition $C = G \circ_g H$ via player $g \in P_G$ by $P_C = (P_G \backslash \{g\}) \cup P_H$ and

$$W_C^{\text{min}} = \{X \subseteq P_C : X \in W_G^{\text{min}}\} \cup \{X \subset P_C : (X \cap P_G) \cup \{g\} \in W_G^{\text{min}} \text{ and } X \cap P_H \in W_H^{\text{min}}\}$$
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eg. Consider the case where $G = H$ are $k$ out of $n$ majority games. Then the minimal winning coalitions of $G \circ_g H$ are those consisting of $k$ players from $G$, or $k - 1$ players from $G$ and $k$ players from $H$. 
Definition

Let $G = (P_G, W_G)$. We define the desirability relation $\preceq_G$ on $G$ by:

$$i \preceq_G j \text{ if for all } U \subseteq P_G \setminus \{i, j\}, \quad U \cup i \in W_G \implies U \cup j \in W_G.$$ 

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- This is a partial ordering.
- Say that a game is complete if ”\( \preceq \)” is a total ordering.
- \( G \) weighted \( \implies \) \( G \) complete.
Definition

A simple game $G$ is swap robust if for any two winning coalitions in that game, say $S$ and $T$, if we swap one player in $S$ with one player in $T$, then the resulting two coalitions are not both losing.
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- $G$ swap robust $\iff G$ complete
- $G$ trade robust $\iff G$ weighted
Theorem

Let $G$ and $H$ be complete games with more than one distinct minimal winning coalition and no dummy players. Then the composition $C = G \circ g H$ is complete if and only if $g$ is a member of the weakest desirability class of $G$. 
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Proof.

$\Leftarrow$: Let $W_1, W_2 \in W_C$. Write $W_1 = X_1 \cup Y_1$ and $W_2 = X_2 \cup Y_2$. $X_i \cup \{g\}$ is winning in $G$ and $Y_i$ is winning in $H$ if $X_i$ is not winning in $G$. Three ways to swap a player from $W_1$ with a player from $W_2$:

1. $x_1 \in X_1$ with $x_2 \in X_2$: $W_1$ or $W_2$ still winning by completeness of $G$.
2. $y_1 \in Y_1$ with $y_2 \in Y_2$: $W_1$ or $W_2$ still winning by completeness of $H$.
3. $x_1 \in X_1$ with $y_2 \in Y_2$ or vice versa: then $X_2 \cup \{x_1\}$ is winning.
Say that a game $G$ is reducible if there exist $G_1, G_2$ such that 
\[
\min\{|P_{G_1}|, |P_{G_2}|\} > 1 \quad \text{such that} \quad G = G_1 \circ_g G_2 \quad \text{for some} \quad g \in G_1.
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Say that a game $G$ is reducible if there exist $G_1, G_2$ such that $\min\{|P_{G_1}|, |P_{G_2}|\} > 1$ such that $G = G_1 \circ_g G_2$ for some $g \in G_1$.

**Theorem**

*The set of all complete games with the operation of composition forms a semigroup. Every complete game can be uniquely decomposed (up to isomorphism) as a composition $G = G_1 \circ_{g_1} G_2 \cdots \circ_{g_{n-1}} G_n$ where each $G_i$ is irreducible.*
Goal: Given weighted voting games $G$ and $H$, and $g \in G$, under what conditions is the composition weighted?
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If $G \circ g H$ is weighted, then $g$ must be (one of) the least desirable player in $G$, or else $G \circ g H$ is not even complete.
Let $G = [7; 3, 3, 2, 2, 2, 2]$ and let $H = [2; 1, 1, 1]$. Label the two players of weight 3 in $G$ as type $A$ players, the players of weight 2 in $G$ as type $B$ players and the players in $H$ as type $C$ players. We have the following certificate of incompleteness for $G \circ_B H$:

$$(AB^2, ABC^2; A^2C, B^3C)$$

So substituting via the least desirable player is not enough to ensure weightedness of the composition.
If $G$ is weighted then we can always find integer weights and quota for $G$. 
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Let $G$ and $H$ be weighted and suppose that there exists an integer system of weights for $G$ such that the least desirable player, $g$, has weight 1. Then $G \circ_g H$ is weighted.
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**Theorem**

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We can prove the theorem by constructing a system of weights for the composition.
Definition (Homogeneous Simple Game)

A homogeneous simple game $G$ is a weighted voting game where it is possible to find a system of weights such that every minimal winning coalition has the same weight.
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- Ostmann (1984) proved that all homogeneous games can be represented by an integer system of weights with some player having weight 1. Thus, if $G$ is homogeneous and $H$ is weighted, then $G \circ_g H$ is weighted.
Open Questions

- Fully characterise conditions for \( G \circ_g H \) to be weighted.
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- Investigate decompositions of arbitrary games.
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- Investigate decompositions of arbitrary games.
- Closure of other classes of game under composition. Eg. Is the composition of two homogeneous games in turn homogeneous?