On the Computation of Fully Proportional Representation
(joint work with Nadja Betzler and Johannes Uhlmann)

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Multi-winner Election Rules

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We will concentrate on the multi-winner rules that solve to some extent the problem of proportional representation (PR).
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There is however a third way forward.

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Decisions of the elected assembly will be made on the basis of their independent judgements but will be as if they reflected the will of people.
The trade-off between inclusiveness and accountability

The key design dilemma is to find a proper system on the following spectrum:

- single-member plurality (SMP) districts, and (accountability 1, inclusiveness 0)
- list systems of proportional representation. (accountability 0, inclusiveness 1)
- proportional representation (PR) through multi-member districts and STV. (accountability $\alpha$, inclusiveness $\beta$)
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What is the best way forward?
Dodgson’s idea

Charles Dodgson (Lewis Carrol) asserted that

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The idea was further advanced by Black (1986), Chamberlin & Courant (1983) and later by Monroe (1995).
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$r$ is a misrepresentation function if:

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In other words, the misrepresentation of $v$ by $c$ is $r_s(v, c) = s_{pos}(c)$, where $s = (s_1, \ldots, s_m)$. 

If $s = (0, 1, 2, \ldots, m-1)$ we call it the Borda misrepresentation function and $s = (0, 1, 1, \ldots, 1)$ is the approval one.
Positional misrepresentation function

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Total (Societal) Misrepresentation

By \( w : V \rightarrow C \) we denote the function that assigns voters to representatives (or the other way around), i.e., under this assignment voter \( v \) is represented by candidate \( w(v) \). The total misrepresentation of the election under \( w \) is then given by

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\sum_{v \in V} r(v, w(v)) \quad \text{or} \quad \max_{v \in V} r(v, w(v))
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in the classical Harsanyi’s and Rawl’s minimax versions.
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Mapping $w$ respects the $M$-criterion if $|w(V)| = k$ and $w$ assigns at least $\lfloor n/k \rfloor$ and at most $\lceil n/k \rceil$ voters to every candidate from $w(V)$. 
They suggested to use Borda misrepresentation function with

\[ s = (0, 1, 2, \ldots, m) \]

and use Harsanyi’s approach to calculate the total misrepresentation. If \( k \) representatives to be elected they look for \( w: V \rightarrow C \) such that \( |w(V)| = k \) and the total misrepresentation is minimized.
Chamberlin-Courant approach

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Requires weighted voting in the elected assembly.
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Monroe’s Fully Proportional Representation

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By using the \( M \)-criterion he avoids assigning weights to representatives in the elected assembly.
Example

Six people have to elect three representative. The profile is:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>a</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>c</td>
<td>b</td>
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</tr>
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</table>

- CC-method elects \( \{ a^2, c \} \) with total misrepresentation 0 \( (a \text{ gets weight 2, } c \text{ gets weight 1}) \);
- M-method elects \( \{ a, b, c \} \) with total misrepresentation 2.
Theorem (Procaccia-Rosenschein-Zohar, 2007)

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In both cases Harsanyi method of calculating the total misrepresentation was used. Can Rawlsian method help?
CC-Multiwinner Problems

\textbf{CC-Multiwinner (CC-MW)}

\textbf{Given:} A set \( C \) of candidates, a multiset \( V \) of voters, a misrepresentation function \( r \), a misrepresentation bound \( R \in \mathbb{Q}_0^+ \) and a positive integer \( k \).

\textbf{Task:} Find a subset \( C' \subseteq C \) of size \( k \) and an assignment of voters \( w \) such that \( w(V) = C' \) and \( \sum_{v \in V} r(v, w(v)) \leq R \).
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**Minimax CC-Multiwinner (Minimax CC-MW)**

**Given:** Same as in CC-Multiwinner.

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The First Result

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Adopting Rawlsian approach does not make computation easier in general.

But we will see that the situation changes completely for single-peaked elections where the minimax version becomes indeed easier.
Parameterized complexity analysis deals with problems which have a distinguished parameter $k$. A problem $P$ is said to be Fixed Parameter Tractable (FPT) if there is an algorithm, that given a pair $(x, k) \in \Sigma^* \times \mathbb{N}$ decides whether or not $(x, k) \in P$ in at most $f(k)|x|^c$ steps, where $f$ is an arbitrary computable function and $c$ does not depend on $k$. 
Parameterized Problems and FPT

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If $(x, k) \in \Sigma^* \times \mathbb{N}$ is an instance of a parameterized problem, we refer to $x$ as the input and $k$ as the parameter.

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steps, where $f$ is an arbitrary computable function and $c$ does not depend on $k$. 
There is a natural hierarchy of parameterized complexity classes

\[ FPT = W[0] \subseteq W[1] \subseteq W[2] \subseteq \ldots \]

intuitively based on the complexity of circuits required to check a solution.

Experimentally shown that \( W[2] \)-complete problems are hard even for small values of the parameter.
Several parameterized reductions in this work are from the W[2]-complete **Hitting Set (HS)** problem:

Given family \( \mathcal{F} = \{ F_1, \ldots, F_n \} \) of subsets over a universe \( U \) and an integer \( k \geq 0 \), decide whether there is a hitting set \( U' \subseteq U \) of size at most \( k \) by which we understand a set \( U' \) such that \( F_i \cap U' \neq \emptyset \) for every \( 1 \leq i \leq n \).

**HS** is NP-hard and W[2]-hard with respect to parameter \( k \) (Fellows-Downey, 1999).
The Hitting Set at work

Minimax CC-Multiwinner for $R = 0$ is exactly the HS. Let $V = V_1 \cup \ldots \cup V_m$ where $V_i$ us the set of voters whose first preference is $c_i$.

Claim. There is a hitting set of size $k$ for $V = \{V_1, \ldots, V_m\}$ if and only if there is a winner set of size $k$ for M-MULTIWINNER that represents all voters with total misrepresentation $R = 0$. 
The misrepresentation function \( r \) is either approval (A), Borda (B) or unrestricted (U).

<table>
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<th>Parameter</th>
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<td>A</td>
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<td>XP</td>
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</tr>
<tr>
<td></td>
<td>B</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
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<tr>
<td># can.</td>
<td>U</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
</tr>
<tr>
<td># vot.</td>
<td>U</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
<td>FPT</td>
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\( R \) and \( k \) must be specified for each parameter.
Results for Single-Peaked Elections

The running times depending on the number $n$ of voters, the number $m$ of candidates, and the number $k$ of winners. If not stated otherwise, the result holds for an arbitrary misrepresentation function.

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M-Multiwinner for the approval misrepresentation function for instances with a single-peaked input profile can be reduced to Max-Hard-1-RS.
**Maximum One-Dimensional Rectangle Stabbing with Hard Constraints (Max-Hard-1-RS)**

**Input:** A set $\mathcal{U} = \{u_1, \ldots, u_n\}$ of horizontal intervals and as set $S = \{S_1, \ldots, S_m\}$ of vertical lines with capacity $c(S) \in \{1, \ldots, n\}$ for every line $S \in S$, and a positive integer $k$.

**Task:** Find a size-$k$ set $S' \subseteq S$ and an assignment $A$ with $|A(S)| \leq c(S)$ for each $S \in S'$ such that $|\bigcup_{S \in S'} A(S)|$ is maximal.

**Theorem**

Maximum One-Dimensional Rectangle Stabbing with Hard Constraints can be solved in $O(n^5 mk^3)$ time.