Power measures derived from the sequential query process

Mark C. Wilson (with R. Reyhani and G. Pritchard)
www.cs.auckland.ac.nz/~mcw/blog/

Department of Computer Science
University of Auckland

3rd CMSS Summer Workshop, 2012-02-21
References

Basic setup

The sequential query process

Semivalues

Application to manipulation measures
Key references

Key references


Key references


TU games

- A transferable utility (TU) game \( G = (X, v) \) is specified by a finite set \( X \) (the players) and a characteristic function \( v : 2^X \rightarrow \mathbb{R} \).
A transferable utility (TU) game $G = (X, v)$ is specified by a finite set $X$ (the players) and a characteristic function $v : 2^X \rightarrow \mathbb{R}$.

Usually, $v(\emptyset) = 0$ is required - we do here.
TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set $X$ (the players) and a characteristic function $v : 2^X \rightarrow \mathbb{R}$.

- Usually, $v(\emptyset) = 0$ is required - we do here.

- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition $S$. 
TU games

- A transferable utility (TU) game $G = (X, v)$ is specified by a finite set $X$ (the players) and a characteristic function $v : 2^X \rightarrow \mathbb{R}$.
- Usually, $v(\emptyset) = 0$ is required - we do here.
- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition $S$.
- Notation:
A transferable utility (TU) game $G = (X, v)$ is specified by a finite set $X$ (the players) and a characteristic function $v : 2^X \rightarrow \mathbb{R}$.

- Usually, $v(\emptyset) = 0$ is required - we do here.

- For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition $S$.

- Notation:
  - $\mathcal{G}$: collection of all TU games - a real vector space
A transferable utility (TU) game $G = (X, v)$ is specified by a finite set $X$ (the players) and a characteristic function $v : 2^X \to \mathbb{R}$.

Usually, $v(\emptyset) = 0$ is required - we do here.

For each $S \subseteq X$, $v(S)$ is supposed to represent the total payoff to the coalition $S$.

Notation:
- $\mathcal{G}$: collection of all TU games - a real vector space
- $\mathcal{G}(X)$: collection of all TU games with underlying set $X$ - a real vector space
Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$. 

- Alternatively, it can be specified by $X$ and the set of winning coalitions, $W := \{S \subseteq X | v(S) = 1\}$.

- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don't.

- Notation:
  - $SG$ : collection of all simple games - a lattice
  - $SG(X)$ : collection of all simple games with underlying set $X$ - a lattice
Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by $X$ and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
Simple games

▶ A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.

▶ Alternatively, it can be specified by $X$ and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.

▶ Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don’t.
Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by $X$ and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don’t.
- Notation:
Simple games

- A simple game is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by $X$ and the set of winning coalitions, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don’t.
- Notation:
  - $SG$: collection of all simple games - a lattice
Simple games

- A **simple game** is a TU game with $v(S) \in \{0, 1\}$ for all $S \subseteq X$.
- Alternatively, it can be specified by $X$ and the set of **winning coalitions**, $W := \{S \subseteq X \mid v(S) = 1\}$.
- Often, $\emptyset \notin W$ is required - we do here. Often $W \neq \emptyset$ is required - we don’t.
- **Notation:**
  - $SG$: collection of all simple games - a lattice
  - $SG(X)$: collection of all simple games with underlying set $X$ - a lattice
Key motivating examples of simple games

- **Unanimity games**: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of $S$ has a veto.
Key motivating examples of simple games

- **Unanimity games**: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of $S$ has a veto.

- **Weighted majority games** $[q; w_1, w_2, \ldots w_n]$. Player $i$ has weight $w_i$; choose a quota $q$ and let $v(S) = 1$ iff $\sum_{i \in S} w_i \geq q$. Used to model yes-no voting in committees. Examples: stockholder elections, EU Council of Ministers, ordinary majority voting in Parliament.
Key motivating examples of simple games

- **Unanimity games**: choose $S \subseteq X$ and let $v(T) = 1$ if and only if $S \subseteq T$. Every member of $S$ has a veto.

- **Weighted majority games** $[q; w_1, w_2, \ldots w_n]$. Player $i$ has weight $w_i$; choose a quota $q$ and let $v(S) = 1$ iff $\sum_{i \in S} w_i \geq q$. Used to model yes-no voting in committees. Examples: stockholder elections, EU Council of Ministers, ordinary majority voting in Parliament.

- **Disequilibrium games**: for a given noncooperative game and fixed profile of actions, declare a subset to be winning if is a witness to the profile not being a strong Nash equilibrium. Examples: voting rules with the sincere profile.
Basic concepts of TU games and simple games

monotonicity \( S \subseteq T \implies v(S) \leq v(T) \).

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.
Basic concepts of TU games and simple games

monotonicity $S \subseteq T \implies v(S) \leq v(T)$.

dummy $i \in X$ is a dummy if for all $S \subseteq X$, $v(S) = v(S \setminus \{i\})$.

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.
Basic concepts of TU games and simple games

monotonicity \( S \subseteq T \implies v(S) \leq v(T) \).

dummy \( i \in X \) is a dummy if for all \( S \subseteq X \),
\[ v(S) = v(S \setminus \{i\}) \]

proper simple game \( S \in W, T \in W \implies S \cap T \neq \emptyset \).

We usually assume monotonicity for simple games, in which case we need only specify the minimal winning coalitions in order to specify the game. A dummy is not an element of any minimal winning coalition.
Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!

- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
- Treaty of Lisbon (from 2014): coalition wins iff it has at least 55% of countries and 65% of population. This method is easily implemented if new members join, and avoids complex negotiations over weights.
Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was $[12; 4, 4, 4, 2, 2, 1]$. Luxembourg was a dummy!
Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was \([12; 4, 4, 4, 2, 2, 1]\). Luxembourg was a dummy!
- Last version (1995–2003) had \([62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]\).
- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
Example: voting in the EU Council of Ministers

- From 1958 to 2003 various weighted majority games were used. Such games are proper if the quota is large enough (more than half the sum of all weights).
- In first version (Treaty of Rome), game was \([12, 4, 4, 4, 2, 2, 1]\). Luxembourg was a dummy!
- Last version (1995–2003) had \([62; 10, 10, 10, 10, 8, 5, 5, 5, 5, 4, 4, 3, 3, 3, 2]\).
- Treaty of Nice (currently in force) uses weights (totalling 345) but has more conditions. A coalition is winning iff it has at least 50% of the countries, 74% of the weights, 62% of the population.
- Treaty of Lisbon (from 2014): coalition wins iff it has at least 55% of countries and 65% of population. This method is easily implemented if new members join, and avoids complex negotiations over weights.
The basic model

- Let \((X, W)\) be a simple game, where \(|X| = n\).
The basic model

- Let $(X, W)$ be a simple game, where $|X| = n$.
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
The basic model

- Let \((X, W)\) be a simple game, where \(|X| = n\).
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.
The basic model

- Let \((X, W)\) be a simple game, where \(|X| = n\).
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.
- Let \(Q\) be the random variable equal to the number of queries in this process, and \(\bar{Q}\) its expectation.
The sequential query process

The basic model

- Let \((X, W)\) be a simple game, where \(|X| = n\).
- Choose (query) elements one at a time uniformly without replacement, until a winning coalition is found.
- This is the same process considered by Shapley and Shubik in defining their famous power index.
- Let \(Q\) be the random variable equal to the number of queries in this process, and \(\bar{Q}\) its expectation.
- If no winning coalition exists, let \(Q\) take the value \(n + 1\).
Another interpretation of $\bar{Q}$

- For $k \in \mathbb{N}$, define the probability measure $m_k$ to be the uniform measure on the set of all subsets of $X$ of size $k$, and let $W_k$ be the set of winning coalitions of size $k$. 

By a standard computation involving tail probabilities, we have

$$Q = n + 1 - \sum_{k=0}^{\lfloor n/2 \rfloor} |W_k| \binom{n}{k}.$$
Another interpretation of $\bar{Q}$

- For $k \in \mathbb{N}$, define the probability measure $m_k$ to be the uniform measure on the set of all subsets of $X$ of size $k$, and let $W_k$ be the set of winning coalitions of size $k$.
- For each $k$ with $0 \leq k \leq n$,

$$\Pr(Q \leq k) = \Pr(W_k)$$

where the latter probability is with respect to $m_k$. 
Another interpretation of $\overline{Q}$

- For $k \in \mathbb{N}$, define the probability measure $m_k$ to be the uniform measure on the set of all subsets of $X$ of size $k$, and let $W_k$ be the set of winning coalitions of size $k$.
- For each $k$ with $0 \leq k \leq n$,

$$\Pr(Q \leq k) = \Pr(W_k)$$

where the latter probability is with respect to $m_k$.
- In other words, the probability that we require at most $k$ queries to find a winning coalition equals the probability that a uniformly randomly chosen $k$-subset is a winning coalition.
Another interpretation of $\bar{Q}$

- For $k \in \mathbb{N}$, define the probability measure $m_k$ to be the uniform measure on the set of all subsets of $X$ of size $k$, and let $W_k$ be the set of winning coalitions of size $k$.
- For each $k$ with $0 \leq k \leq n$,
  \[ \Pr(Q \leq k) = \Pr(W_k) \]
  where the latter probability is with respect to $m_k$.
- In other words, the probability that we require at most $k$ queries to find a winning coalition equals the probability that a uniformly randomly chosen $k$-subset is a winning coalition.
- By a standard computation involving tail probabilities, we have
  \[ \bar{Q} = n + 1 - \sum_{k=0}^{n} \frac{|W_k|}{\binom{n}{k}}. \]
Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say $F$ is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$. 
Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say $F$ is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$.
- There is a bijection $F \leftrightarrow f$ given by

$$f(n, k) = \frac{F(n, k) - F(n, k + 1)}{\binom{n}{k}}$$

Note that $F$ is admissible if and only if $f$ is nonnegative and

$$\sum_{k=0}^{n} f(n, k) \binom{n}{k} = 1.$$
Changing variables

- Let $F : \mathbb{N}^2 \to \mathbb{R}$. Say $F$ is an admissible change of variables if $F(n, \cdot)$ is decreasing, $F(n, 0) = 1$ and $F(n, k) = 0$ whenever $k > n$.

- There is a bijection $F \leftrightarrow f$ given by

$$f(n, k) = \frac{F(n, k) - F(n, k + 1)}{\binom{n}{k}}$$

Note that $F$ is admissible if and only if $f$ is nonnegative and

$$\sum_{k=0}^{n} f(n, k) \binom{n}{k} = 1.$$

- There is a bijection $F \leftrightarrow \mu$ given by

$$\mu(n, j) = F(n, k) - F(n, k + 1)$$

Note that $F$ is admissible if and only if for each $n$, $\mu(n, \cdot)$ is a probability measure on $\{0, \ldots, n\}$. 
Define $Q^*_F : SG \rightarrow \mathbb{R}$ by

$$Q^*_F(G) = E[F(Q)]$$

where the expectation is taken as above.
The sequential query process

Define $Q^*_F : SG \to \mathbb{R}$ by

$$Q^*_F(G) = E[F(Q)]$$

where the expectation is taken as above.

We have

$$Q^*_F = \sum_{k=0}^{n} f(n, k)|W_k|$$
Define \( Q_F^* : S\mathcal{G} \to \mathbb{R} \) by

\[
Q_F^*(G) = E[F(Q)]
\]

where the expectation is taken as above.

We have

\[
Q_F^* = \sum_{k=0}^{n} f(n, k) |W_k|
\]

There is an obvious generalization to TU-games:

\[
Q_F^*(G) = \sum_{k=0}^{n} f(n, k) \sum_{|S|=k, S \subseteq X} v(S) = \sum_{S \subseteq X} f(n, |S|)v(S).
\]
Properties of $Q^*_F$

$Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:

1. **Anonymity**: depends only on the isomorphism class of the game.
2. **Positivity**: is nonnegative on monotone games.
3. **Dummy**: adding a null player (not a member of any winning coalition) has no effect.
4. **Regularity**: is strictly positive unless the game has no winning coalitions.

**Special cases:**

- Choosing $f(n,k) = 2^n$ yields the Coleman index.
- For self-dual (strong and proper) games, $Q^*_F = \frac{1}{2}$.
- For the weighted majority game with quota $q$, $Q^*_F = F(n,q)$. 
Properties of $Q^*_F$

- $Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
Properties of $Q^*_F$

$Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:

- **Anonymity**: depends only on the isomorphism class of the game.
- **Positivity**: is nonnegative on monotone games.
Properties of $Q^*_F$

- $Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
Properties of $Q^*_F$

$Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:

- **Anonymity:** depends only on the isomorphism class of the game.
- **Positivity:** is nonnegative on monotone games.
- **Dummy:** adding a null player (not a member of any winning coalition) has no effect.
- **Regularity:** is strictly positive unless the game has no winning coalitions.

Special cases:

- Choosing $f(n,k) = 2^{-n}$ yields the Coleman index.
- For self-dual (strong and proper) games, $Q^*_F = 1/2$.
- For the weighted majority game with quota $q$, $Q^*_F = F(n,q)$. 
Properties of $Q^*_F$

- $Q^*_F$ is a decisiveness index on $SG(X)$. It satisfies:
  - **Anonymity**: depends only on the isomorphism class of the game.
  - **Positivity**: is nonnegative on monotone games.
  - **Dummy**: adding a null player (not a member of any winning coalition) has no effect.
  - **Regularity**: is strictly positive unless the game has no winning coalitions.
- **Special cases**:
  - Choosing $f(n, k) = 2^n$ yields the Coleman index.
  - For self-dual (strong and proper) games, $Q^*_F = \frac{1}{2}$.
  - For the weighted majority game with quota $q$, $Q^*_F = F(n, q)$. 

Mark C. Wilson
Properties of $Q^*_F$

- $Q^*_F$ is a **decisiveness index** on $SG(X)$. It satisfies:
  - **Anonymity**: depends only on the isomorphism class of the game.
  - **Positivity**: is nonnegative on monotone games.
  - **Dummy**: adding a null player (not a member of any winning coalition) has no effect.
  - **Regularity**: is strictly positive unless the game has no winning coalitions.

- **Special cases**:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.
Properties of $Q^*_F$

- $Q^*_F$ is a **decisiveness index** on $SG(X)$. It satisfies:
  - Anonymity: depends only on the isomorphism class of the game.
  - Positivity: is nonnegative on monotone games.
  - Dummy: adding a null player (not a member of any winning coalition) has no effect.
  - Regularity: is strictly positive unless the game has no winning coalitions.

- Special cases:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.
  - For self-dual (strong and proper) games, $Q^*_F = 1/2$. 
Properties of $Q_F^*$

- $Q_F^*$ is a **decisiveness index** on $SG(X)$. It satisfies:
  - **Anonymity**: depends only on the isomorphism class of the game.
  - **Positivity**: is nonnegative on monotone games.
  - **Dummy**: adding a null player (not a member of any winning coalition) has no effect.
  - **Regularity**: is strictly positive unless the game has no winning coalitions.

- **Special cases**:
  - Choosing $f(n, k) = 2^{-n}$ yields the Coleman index.
  - For self-dual (strong and proper) games, $Q_F^* = 1/2$.
  - For the weighted majority game with quota $q$, $Q_F^* = F(n, q)$. 

Mark C. Wilson
Values and semivalues

- A value is a function $G \rightarrow AG$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.

Dubey, Neyman and Weber (1981) showed that a value is a semivalue if and only if it has the form

$$\xi_i(G) = \sum_{k=0}^{n} p(n,k) \sum_{|S|=k, S \subseteq X} \left[ v(S) - v(S \{i\}) \right]$$

where $p(n,k) \geq 0$ and the following identities hold

$$\sum_{k} \binom{n-1}{k-1} p(n,k) = 1$$
$$p(n,k) = p(n-1,k-1) - p(n,k-1)$$

If all $p(n,k) \neq 0$, the semivalue is called regular.
Values and semivalues

- A value is a function $G \rightarrow AG$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.

- Dubey, Neyman and Weber (1981) showed that a value is a semivalue if and only if it has the form

$$\xi_i(G) = \sum_{k=0}^{n} p(n, k) \sum_{|S| = k, S \subseteq X} [v(S) - v(S \setminus \{i\})]$$

where $p(n, k) \geq 0$ and the following identities hold

$$\sum_k \binom{n-1}{k-1} p(n, k) = 1$$

$$p(n, k) = p(n - 1, k - 1) - p(n, k - 1)$$
Values and semivalues

- A value is a function $G \rightarrow AG$. Those satisfying Anonymity, Dummy, Positivity and Linearity are called semivalues.
- Dubey, Neyman and Weber (1981) showed that a value is a semivalue if and only if it has the form

$$
\xi_i(G) = \sum_{k=0}^{n} p(n, k) \sum_{|S|=k, S \subseteq X} [v(S) - v(S \setminus \{i\})]
$$

where $p(n, k) \geq 0$ and the following identities hold

$$
\sum_{k} \binom{n-1}{k-1} p(n, k) = 1
$$

$$
p(n, k) = p(n - 1, k - 1) - p(n, k - 1)
$$

- If all $p(n, k) \neq 0$, the semivalue is called regular.
Famous semivalues include the Shapley and Banzhaf values, corresponding to \( p(n, k) = [k \binom{n}{k}]^{-1} \) and \( p(n, k) = 2^{1-n} \) respectively.
Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to \( p(n, k) = [k \binom{n}{k}]^{-1} \) and \( p(n, k) = 2^{1-n} \) respectively.

- Every semivalue is uniquely determined by its value on unanimity games.
Semivalues

- Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n, k) = \left[ k \binom{n}{k} \right]^{-1}$ and $p(n, k) = 2^{1-n}$ respectively.

- Every semivalue is uniquely determined by its value on unanimity games.

- Regular semivalues satisfy many nice properties, such as Young sensibility: if the marginal contribution to each $S$ is strictly higher in one game than another, then the $\xi_i$ have the same relation.
Famous semivalues include the Shapley and Banzhaf values, corresponding to $p(n, k) = [k \binom{n}{k}]^{-1}$ and $p(n, k) = 2^{1-n}$ respectively.

Every semivalue is uniquely determined by its value on unanimity games.

Regular semivalues satisfy many nice properties, such as Young sensibility: if the marginal contribution to each $S$ is strictly higher in one game than another, then the $\xi_i$ have the same relation.

Almost all “power measures” in the literature are semivalues. The class of probabilistic values is even more general - the coefficients $p$ can depend on $S$ and not just on $|S|$.
Semivalues and coalition formation models

Consider the following model of coalition formation: fix a probability distribution on $2^X$, assume that each possible coalition (subset $S$ of $X$) forms with probability $p(S)$, and that only one coalition $S$ will form.

The ex ante expected marginal contribution of $i$ to $S$ is $E[D_i(S)] := E[v(S) - v(S\{i\})] = \sum_{S: i \in S} p(S)(v(S) - v(S\{i\}))$.

The ex interim expected marginal contribution of $i$ to $S$, conditional on $i \in S$, is $\Phi_i(v,p) := E[D_i(S) | S \ni i] = E[D_i(S)] \Pr(S \ni i)$.

There is a bijection $p \leftrightarrow \Phi(\cdot, p)$. 

Mark C. Wilson
Semivalues and coalition formation models

Consider the following model of coalition formation: fix a probability distribution on $2^X$, assume that each possible coalition (subset $S$ of $X$) forms with probability $p(S)$, and that only one coalition $S$ will form.

The \textit{ex ante} expected marginal contribution of $i$ to $S$ is

$$E[D_i(S)] := E[v(S) - v(S \setminus \{i\})] = \sum_{S : i \in S} p(S)(v(S) - v(S \setminus \{i\})).$$
Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on $2^X$, assume that each possible coalition (subset $S$ of $X$) forms with probability $p(S)$, and that only one coalition $S$ will form.

- The \textit{ex ante} expected marginal contribution of $i$ to $S$ is

$$E[D_i(S)] := E[v(S) - v(S \setminus \{i\})] = \sum_{S:i \in S} p(S) (v(S) - v(S \setminus \{i\})).$$

- The \textit{ex interim} expected marginal contribution of $i$ to $S$, conditional on $i \in S$, is

$$\Phi_i(v, p) := E[D_i(S) \mid S \ni i] = \frac{E[D_i(S)]}{\Pr(S \ni i)}.$$
Semivalues and coalition formation models

- Consider the following model of coalition formation: fix a probability distribution on $2^X$, assume that each possible coalition (subset $S$ of $X$) forms with probability $p(S)$, and that only one coalition $S$ will form.

- The *ex ante* expected marginal contribution of $i$ to $S$ is

$$
E[D_i(S)] := E[v(S) - v(S \setminus \{i\})] = \sum_{S : i \in S} p(S) (v(S) - v(S \setminus \{i\})).
$$

- The *ex interim* expected marginal contribution of $i$ to $S$, conditional on $i \in S$, is

$$
\Phi_i(v, p) := E[D_i(S) \mid S \ni i] = \frac{E[D_i(S)]}{\Pr(S \ni i)}.
$$

- There is a bijection $p \leftrightarrow \Phi(\cdot, p)$. 

Mark C. Wilson
Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.

\[ \Phi(G) - \Phi(G - \{i\}) = \xi_i(G) \]

for all \( G = (X, v) \in G \) such that \( X \neq \emptyset \). Here \( G - \{i\} \) is the game with player set \( X \{i\} \) and the same \( v \).

The initial condition \( \Phi(\emptyset, v) = 0 \) is usually assumed.

There is a unique efficient value having a potential function, and it is the Shapley value. Explicitly, the potential looks like

\[ \Phi(G) = \frac{1}{n!} \sum_{|S| = k, S \subseteq X} v(S) \]
Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let \( \xi \) be a value. A potential for \( \xi \) is a mapping \( \Phi : \mathcal{G} \to \mathbb{R} \) such that
  \[
  \Phi(G) - \Phi(G-\{i\}) = \xi_i(G)
  \]
  for all \( G = (X,v) \in \mathcal{G} \) such that \( X \neq \emptyset \). Here \( G-\{i\} \) is the game with player set \( X \setminus \{i\} \) and the same \( v \).
Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let $\xi$ be a value. A potential for $\xi$ is a mapping $\Phi : \mathcal{G} \to \mathbb{R}$ such that
  \[
  \Phi(G) - \Phi(G - \{i\}) = \xi_i(G)
  \]
  for all $G = (X, v) \in \mathcal{G}$ such that $X \neq \emptyset$. Here $G - \{i\}$ is the game with player set $X \setminus \{i\}$ and the same $v$.
- The initial condition $\Phi(\emptyset, v) = 0$ is usually assumed.
Potential

- Mas-Colell and Hart (1988) introduced the idea of potential, borrowed from physics.
- Let $\xi$ be a value. A potential for $\xi$ is a mapping $\Phi : \mathcal{G} \to \mathbb{R}$ such that
  \[ \Phi(G) - \Phi(G_{-\{i\}}) = \xi_i(G) \]
  for all $G = (X, v) \in \mathcal{G}$ such that $X \neq \emptyset$. Here $G_{-\{i\}}$ is the game with player set $X \setminus \{i\}$ and the same $v$.
- The initial condition $\Phi(\emptyset, v) = 0$ is usually assumed.
- There is a unique efficient value having a potential function, and it is the Shapley value. Explicitly, the potential looks like
  \[ \Phi(G) = \sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} \sum_{|S|=k, S \subseteq X} v(S). \]
Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.

\[ \xi_i(G) - \xi_i(G\{j\}) = \xi_j(G) - \xi_j(G\{i\}) \]

and if and only if it is path-independent.

In particular, every semivalue has a potential function.

Explicitly:

\[ \Phi(G) = \sum_k p(n,k) \sum_{|S| = k} v(S) \]
Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.
- The answer: $\xi$ has a potential if and only if it satisfies Myerson’s **balanced contributions axiom**:

$$
\xi_i(G) - \xi_i(G \setminus \{j\}) = \xi_j(G) - \xi_j(G \setminus \{i\})
$$

and if and only if it is **path-independent**.
Potential without efficiency

- Calvo and Santos (2000) described exactly which values possess a potential function.
- The answer: $\xi$ has a potential if and only if it satisfies Myerson’s balanced contributions axiom:

$$\xi_i(G) - \xi_i(G \setminus \{j\}) = \xi_j(G) - \xi_j(G \setminus \{i\})$$

and if and only if it is path-independent.
- In particular, every semivalue has a potential function. Explicitly:

$$\Phi(G) = \sum_k p(n, k) \sum_{|S|=k} v(S)$$
The marginal function

- It is readily shown that $Q_F^*$ is the potential function of a function $q_F^*$, given by

$$q_{F,i}^* = \sum_{S:\i \in S} f(n, |S|) D_i(S)$$
The marginal function

- It is readily shown that $Q^*_F$ is the potential function of a function $q^*_F$, given by

$$q^*_F,i = \sum_{S:i \in S} f(n, |S|)D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).
The marginal function

- It is readily shown that $Q^*_F$ is the potential function of a function $q^*_F$, given by

$$q^*_F,i = \sum_{S:i \in S} f(n, |S|)D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).
- There is a bijection between probability measures on $\{0, 1, \ldots, n\}$ and weighted semivalues on $G_n$ given by $\mu_n \leftrightarrow q^*_F$. 
The marginal function

- It is readily shown that $Q^*_F$ is the potential function of a function $q^*_F$, given by

$$q^*_{F,i} = \sum_{S:i \in S} f(n, |S|) D_i(S)$$

- Such a function is a weighted semivalue (satisfies all properties except the normalization condition).

- There is a bijection between probability measures on $\{0, 1, \ldots, n\}$ and weighted semivalues on $\mathcal{G}_n$ given by $\mu_n \leftrightarrow q^*_F$.

- Under the coalition formation model above, $q^*_{F,i}$ describes the ex ante expected contribution of $i$ to $S$, while the semivalue obtained by normalizing gives the ex interim expected marginal contribution of $i$ to $S$, conditional on $i \in S$. 
The simplest functional form

- The choice $F(n, k) = 1 - \frac{k}{n+1}$ is the simplest form for $F$. It corresponds to $f(n, k) = \frac{1}{(n+1)\binom{n}{k}}$. 

Mark C. Wilson
The simplest functional form

- The choice $F(n, k) = 1 - \frac{k}{n+1}$ is the simplest form for $F$. It corresponds to $f(n, k) = \frac{1}{(n+1)\binom{n}{k}}$. 
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.
The simplest functional form

- The choice \( F(n, k) = 1 - \frac{k}{n+1} \) is the simplest form for \( F \). It corresponds to \( f(n, k) = \frac{1}{(n+1)\binom{n}{k}} \).
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.
- It yields a new decisiveness index, which we call \( Q_0^* \).
The simplest functional form

- The choice \( F(n, k) = 1 - \frac{k}{n+1} \) is the simplest form for \( F \). It corresponds to \( f(n, k) = \frac{1}{(n+1)\binom{n}{k}} \).
- This corresponds to the coalition formation model in which we choose a coalition size uniformly, and then a coalition of that size uniformly.
- It yields a new decisiveness index, which we call \( Q^*_0 \).
- The sequential interpretation is that we query elements one by one until we find a winning coalition, and score 1 for each query.
Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.
Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.

- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.

- Social choice theorists have tried to measure manipulability in many ways, most of them rather crude. There has been no definition of what such a measure should be, and no desirable axioms listed.
Manipulability measures

- Gibbard-Satterthwaite implies that (almost) every social choice function allows a nonempty simple manipulation game for some preference profile.
- The simple game describing manipulability is complicated: can fail to be weighted, strong, proper, or nonempty.
- Social choice theorists have tried to measure manipulability in many ways, most of them rather crude. There has been no definition of what such a measure should be, and no desirable axioms listed.
- Measures found in the literature include: indicator of winning coalition of size 1; number of winning coalitions of size 1; minimum size of a manipulating coalition.
Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable $F$ we can have any decisiveness index we like.
Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.

- Using the query model as above, by choosing a suitable $F$ we can have any decisiveness index we like.

- We think that $Q_0$ is a good candidate, because of its simplicity in terms of the sequential query model.
Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.
- Using the query model as above, by choosing a suitable $F$ we can have any decisiveness index we like.
- We think that $Q_0$ is a good candidate, because of its simplicity in terms of the sequential query model.
- If each voter can have a different cost to recruit (as in bribery), a TU (cost) game is more appropriate than a simple game, but similar ideas should work.
Manipulation measures and query model

- Idea: use a collective decisiveness measure on the associated disequilibrium game to measure the ease of manipulation of a given profile. This allows a principled choice of measure for a given situation, each rooted in a model of coalition formation.

- Using the query model as above, by choosing a suitable $F$ we can have any decisiveness index we like.

- We think that $Q_0$ is a good candidate, because of its simplicity in terms of the sequential query model.

- If each voter can have a different cost to recruit (as in bribery), a TU (cost) game is more appropriate than a simple game, but similar ideas should work.

- Bachrach, Elkind and Faliszewski have used a closely related TU framework to study manipulation of voting rules.
Open problems

- Unify recent results on complexity and power indices (e.g. Faliszewski and coauthors) and generalize them to the case of (regular) semivalues.
Open problems

- Unify recent results on complexity and power indices (e.g. Faliszewski and coauthors) and generalize them to the case of (regular) semivalues.