Best reply dynamics for scoring rules

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Player (voter) action is to submit an *expressed vote* (possibly different from its *sincere preference*).

We are interested in the equilibrium result to predict the winner of election.

Gibbard-Satterthwaite and other theorems show that dominant strategies don’t always exist.

The common solution concept is Nash Equilibrium (NE).

Far too many NE exist and some of them are trivial. Also, voters cannot coordinate on one equilibrium.
We can use learning models or dynamic process (iterative games) as a coordination device.

Fudenberg and Levine (1998): "In some cases, most learning models do not converge to any equilibrium and just coincide with the notion of rationalizability."

However, if it converges, it necessarily finds a NE.

It has application for voting in finding consensus: doodle.com, etc.
In the voting case it has only been used by Meir et al, for best-reply dynamics (BRD) as far as we know (AAAI 2010).

The first player moves, then another one responds, etc

Players move one at a time and in each move, they do a best reply.

Myopic moves, no communication between players and zero knowledge of others.
We have a set $C$ of alternatives (candidates) and set $V$ of voters, with $m := |C|$, $n := |V|$.

Each voter $v$ submits a permutation $L(v)$ of the candidates. This defines the set $T$ of types, and $|T| = m!$.

A profile is a function $V \rightarrow T$. A voting situation is a multiset from $T$ with total weight $n$.

The scoring rule determined by a vector $w$ with $w_1 \geq w_2 \geq \cdots \geq w_{m-1} \geq w_m$ assigns the score

$$|c| := \sum_{t \in T} |\{v \in V \mid L(v) = t\}| w_{L(v)^{-1}(c)}.$$

Special cases:

- plurality: $w = (1, 0, 0, \ldots, 0)$;
- antiplurality (veto): $w = (1, 1, \ldots, 1, 0)$;
- Borda: $w = (m-1, m-2, \ldots, 1, 0)$. 

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Consider antiplurality system with 2 voters $\mathcal{V} = \{1, 2\}$ and 4 candidates $\mathcal{C} = \{a, b, c, d\}$, alphabetically tie-breaking. The sincere profile is $P_0 = (acbd, bacd)$. If voters start from sincere state, we have

\[
(-d, -d)\{a\} \xrightarrow{2} (-d, -a)\{b\} \xrightarrow{1} (-b, -a)\{c\} \xrightarrow{2} (-b, -c)\{a\}
\]

The best reply is not unique. For example, the last move by second player can instead be $-d$. However, $-c$ (vetoing the current winner) is what we call Restricted Best Reply (RBR) for antiplurality.
Summary of Meir et al

- Plurality voting
- Other assumptions:
  - Behaviour: $\text{RBR}$ at each step or best reply.
  - Indifference: keep last move or truth-biased.
  - Initial state: sincere profile or arbitrary profile.
  - Tiebreaking: deterministic or uniform random.
  - Voters: unweighted or weighted.

- Convergence for plurality under red hypotheses in at most $m^2n^2$ steps.
Results:

- Deterministic tiebreaking from an arbitrary initial state converges for unweighted voters.
- The winner is the sincere winner or a candidate at most 1 point behind initially.
- In each case, changing each red hypothesis and keeping the others yields examples of non-convergence.
Example

Consider the sincere profile $P_0 = (abc, bca)$ and voting rule Borda and alphabetical tie-breaking.

\[
\begin{align*}
(abc, bca)\{b\} & \xrightarrow{1} (acb, bca)\{a\} & \xrightarrow{2} (acb, cba)\{c\} & \xrightarrow{1} \\
(abc, cba)\{a\} & \xrightarrow{2} (abc, bca)\{b\} & \diamond
\end{align*}
\]
Best reply is not unique, because several preference orders may yield the same result.
Is there a natural restricted best reply which is unique?
One answer can be the best reply that maximizes the winning score margin of the new winner.
Without loss of generality, we can assume that 
\[ S_i = \{-c | c \in C\} \]

\( o_t \) is the winner set after the move of player \( i \) at time \( t \) and is the current winner set.

Best reply: 
\[ -a \rightarrow -b \text{ where } b \neq a. \]

Type 1: \( a \notin o_t \) and \( b \in o_{t-1} \)

Type 2: \( a \in o_t \) and \( b \notin o_{t-1} \)

Type 3: \( a \in o_t \) and \( b \in o_{t-1} \)

For plurality and antiplurality, allowing type 2 moves can lead to a cycle. We call type 1 and 3 as **RBR**.

In randomized tie-breaking, the player vetoes the least preferred member of \( o_{t-1} \).
RBRD for antiplurality
Set of potential winners

**Definition**

(set of potential winners) The set of potential winners at time $t$, $W_t$, those candidates who have a chance of winning at the next step (time $t + 1$), depending on the different RBR of voters.

$$W_t = \{ c \mid \text{if a player moves } -c \to -b \text{ at time } t + 1, \text{ then } c \in o_{t+1} \}$$
Lemma

If $t < t'$ then $W_t \subseteq W'_t$.

Proof.

- Let $c \in W_{t-1}$, and we have an improvement step $-a \rightarrow -b$ at time $t$. Then we show $c \in W_t$.
- If it is type 3, it is easy to show $-c \rightarrow -a$ makes $c$ winner.
- If it is a type 1, let $b' = o_t$. Note that $b' \notin \{a, b\}$. Then we show $-c \rightarrow -b'$ makes $c$ winner. It uses the transitivity of lexicographic order which may not be true for an arbitrary deterministic tie-breaking rule.
Lemma

Each voter has at most one type 1 move and at most \( m - 1 \) moves of type 3.

Proof.

1. Suppose a step \(-a \xrightarrow{i} b\) is a type 1 move by voter \( i \) at time \( t \). With proof by contradiction, we show this improvement step is the first improvement step of voter \( i \).

2. As at every step \(-a \xrightarrow{i} b\) of type 3, it must hold that \( a \succ_i b \) because of the definition of improvement step.

Conclusion: RBRD for \( G(V, C, A) \) with alphabetical tie-breaking will converge to a NE from any state in at most \( mn \) steps.
Definition

(Stochastic dominance improvement step) Voter $v$ prefers an outcome with winner set $W$ to an outcome with winner set $W'$ is preferred to $W'$ if and only if for each $k = 1 \cdots m$, the probability of electing one of the first $k$ candidates given outcome $W$ should be no less than given $W'$.

Lemma

If $t < t'$ then $W_t \subseteq W_t'$. 
Lemma

Each voter has at most one type 1 move and at most \( m - 1 \) moves of type 3.

Proof.

For type 1 move, similarly proof by contradiction. For type 3 move \(-a \rightarrow -b\), we show the probability of winning of \( a \) has increased and \( b \) has decreased. Therefore, \( a \succ_i b \).

Conclusion: Stochastic dominance RBRD for \( G(\mathcal{V}, \mathcal{C}, A) \) with randomized tie-breaking will converge to a NE from any state in at most \( mn \) steps.
RBRD for plurality

Type 1: \( a \notin o_{t-1} \) and \( b \in o_t \)
Type 3: \( a \in o_{t-1} \) and \( b \in o_t \)

\[ W_t = \{ c \mid \text{if a player moves } a \rightarrow c \text{ and } a \in o_{t-1} \text{ then } c \in o_t \} \]

**Lemma**

If \( t < t' \) then \( W'_t \subseteq W_t \).

**Lemma**

The number of type 1 moves are at most \( m \) and each voter has at most \( m - 1 \) moves of type 3.

**Theorem**

RBRD \( G(V, C, P) \), will converge to a NE from any state in at most \( m + (m - 1)n \) steps.
Cycles for more general scoring rules

Cycle for scoring rules close to Plurality:

- Suppose we have 3 candidates $a$, $b$ and $c$ and $P_0 = (abc, bca)$. The scoring rule is $w = (1, \alpha, 0); \alpha \leq \frac{1}{2}$ and we use alphabetical tie-breaking.

$$
(abc, bca)\{b\} \rightarrow (acb, bca)\{a\} \rightarrow (acb, cba)\{c\} \rightarrow (abc, bca)\{b\}
$$

- Suppose we have 3 candidates $a$, $b$ and $c$ and $P_0 = (abc, bca)$. The scoring rule is $w = (1, \alpha, 0); \alpha \leq \frac{1}{2}$ and we use alphabetical tie-breaking.

$$
(abc, bca)\{a\} \rightarrow (abc, cba)\{b\} \rightarrow (abc, bca)\{a\}
$$

- general $m$ and $n = 2$

$$
(ab\cdots c, bc\cdots a)\{b\} \rightarrow (a\cdots cb, bc\cdots a)\{a\} \rightarrow (ab\cdots c, cb\cdots a)\{a\} \rightarrow (ab\cdots c, bc\cdots a)\{b\}
$$
Cycle for scoring rules close to antiplurality: \( m = 3, n = 4 \)

Suppose we have 3 candidates \( a, b \) and \( c \). The sincere profile is \( P_0 = (abc, bac, cab, bca) \). Our scoring rule is \((1, \alpha, 0); \frac{1}{2} \leq \alpha < 1 \) with alphabetical tie-breaking.

\[
\begin{align*}
(abc, bac, cab, bca)\{b\} & \xrightarrow{1} (acb, bac, cab, bca)\{a\} \xrightarrow{4} (abc, bac, cab, bca)\{b\} \\
(acb, bac, cab, cba)\{c\} & \xrightarrow{1} (acb, bac, cab, cba)\{a\} \xrightarrow{4} (abc, bac, cab, bca)\{b\}
\end{align*}
\]
Consider Borda system with 4 voters and 3 candidates, $P_0 = (acb, acb, cab, cba)$ and alphabetically tie-breaking

$P$ is the same as $P$ after fourth move and the cycle starts.

Now let’s consider another order for the players.

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The possibility of winning of a candidate depends on the type of improvement step and also candidate’s priority in tie-breaking ($\max d = 2$).

The number of type 2 moves are not bounded, so we need to use RBR for convergence.

We need to use stochastic dominance RBR for randomized tie-breaking.

The results of convergence do not happen for a non-linear deterministic tie-breaking rule.

The order of players matters in convergence, equilibrium result and also speed of convergence.